Exact solutions of the wave equation with complex source locations

Richard W. Ziolkowski
Lawrence Livermore National Laboratory, P.O. Box 808, L-156, Livermore, California 94550

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New exact solutions of the homogeneous, free-space wave equation are obtained. They originate from complex source points moving at a constant rate parallel to the real axis of propagation and, therefore, they maintain a Gaussian profile as they propagate. Finite energy pulses can be constructed from these Gaussian pulses by superposition.

I. INTRODUCTION

A recent article by Brittingham[1] has indicated the existence of a new type of solution of the homogeneous, free-space wave equation

$$\Box\Phi(\mathbf{r},t) = \{\Delta - \partial_{ct}^2\}\Phi(\mathbf{r},t) = 0, \tag{1}$$

which was termed a focus wave mode (FWM) for its alleged solitonlike properties. It has been found that this FWM is but one of a class of solutions of (1). In particular, assuming the desired direction of propagation is along the z axis, a solution of the form

$$\Phi(\mathbf{r},t) = e^{ik(z+ct)}F_k(x,y,z-ct) \tag{2}$$

reduces (1) to a Schrödinger equation; i.e.,

$$e^{-ik(z+ct)}\Box e^{ik(z+ct)}F_k = \{\Delta_\perp + 4ik\,\partial_\tau\}F_k = 0, \tag{3}$$

where the transverse Laplacian is $\Delta_\perp = \partial_x^2 + \partial_y^2$ and the characteristic variables are $(\tau,\sigma) = (z-ct, z+ct)$. Equation (2) has a symmetric solution ($\rho^2 = x^2 + y^2$)

$$F_k(x,y,\tau) = \exp[-k\rho^2/(z_0 + i\tau)]/4\pi i(z_0 + i\tau), \tag{4}$$

that originates at the complex source location $(\rho,\tau) = (0, iz_0)$,

$$\{\Delta_\perp + 4ik\,\partial_\tau\}F_k(x,y,\tau) = \delta(\boldsymbol{\rho})\delta(\tau - iz_0), \tag{5}$$

the right-hand side being identically zero for a point in real space-time.

Clearly, the source location $z = iz_0 + ct$ moves parallel to the real z axis. Moreover, defining the complex variance $V = z_0 + i\tau$ so that

$$1/V = 1/A - i/R, \tag{6}$$

it is recognized immediately that (4) represents a moving Gaussian beam with beam spread $A = z_0^2 + (\tau^2/z_0)$, phase front curvature $R = \tau + (z_0^2/\tau)$, and normalized beam waist $(A/k)^{1/2}$. Consequently, combining (2) and (4), the fundamental solution

$$\Phi(\mathbf{r},t) = \phi_k(\rho,\tau,\sigma) = \frac{\exp[ik\sigma - k\rho^2/(z_0 + i\tau)]}{4\pi i(z_0 + i\tau)} \tag{7}$$

is a modulated, moving Gaussian pulse.

II. SOLUTION BEHAVIOR

A short time history of $\mathrm{Re}\,\Phi$ with $f = kc/2\pi = 3.0\times 10^9$ and $z_0 = 1.0$ is shown in Fig. 1. The second subplots are contour plots of the three-dimensional surface plots given in the first subplot. The Gaussian profile of the pulse is apparent. Notice that this profile is maintained during propagation with only local variations. The latter occur primarily near the profile center $(\rho,\tau) = (0,0)$. The variation in the shape of Φ with k is illustrated in Fig. 2. The pulse is concentrated near the ρ axis for small k and becomes more concentrated along the z axis for large k. The unusual features of the plots in Fig. 2(c) such as the jagged peaks and the ragged contours are artifacts of the coarseness of the computational grid. It has also been demonstrated that the pulse amplitude decreases as z_0 increases.

The FWM solution of Maxwell's equations in Ref. 1 is readily obtained from (7) with a Hertzian potential formula-

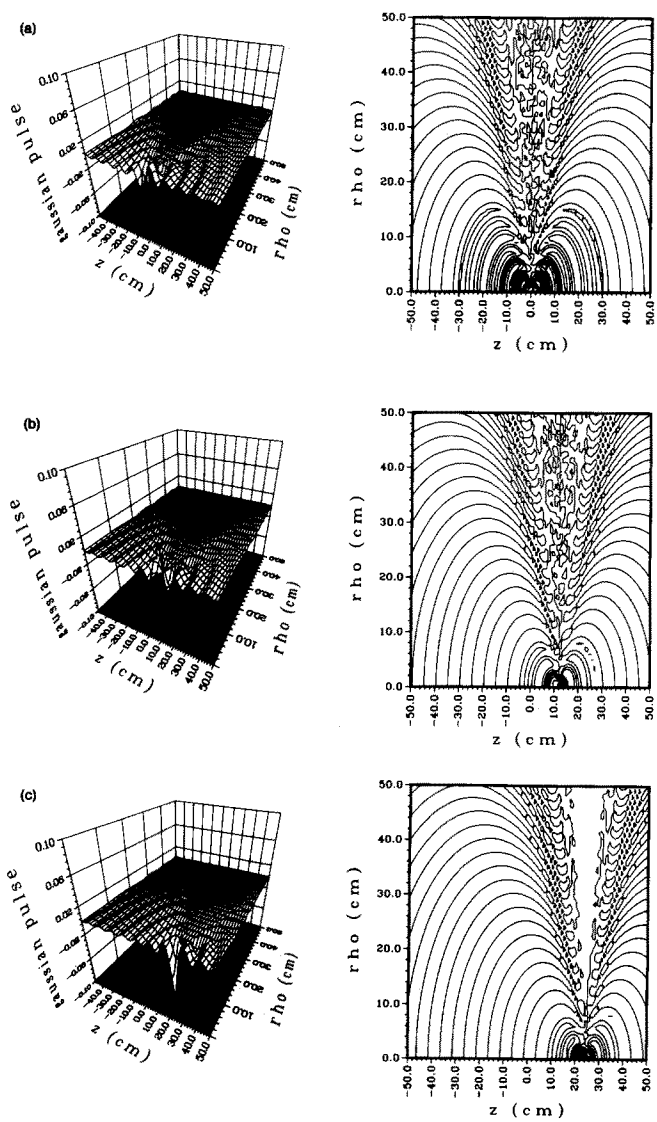

FIG. 1. A time sequence of the fundamental solution (7) for $f = kc/2\pi = 3.0\times 10^9$ and $z_0 = 1.0$ demonstrates that it is a modulated moving Gaussian pulse: (a) $t = 0.0$, (b) $t = 4.0\times 10^{-10}$, (c) $t = 8.0\times 10^{-10}$.

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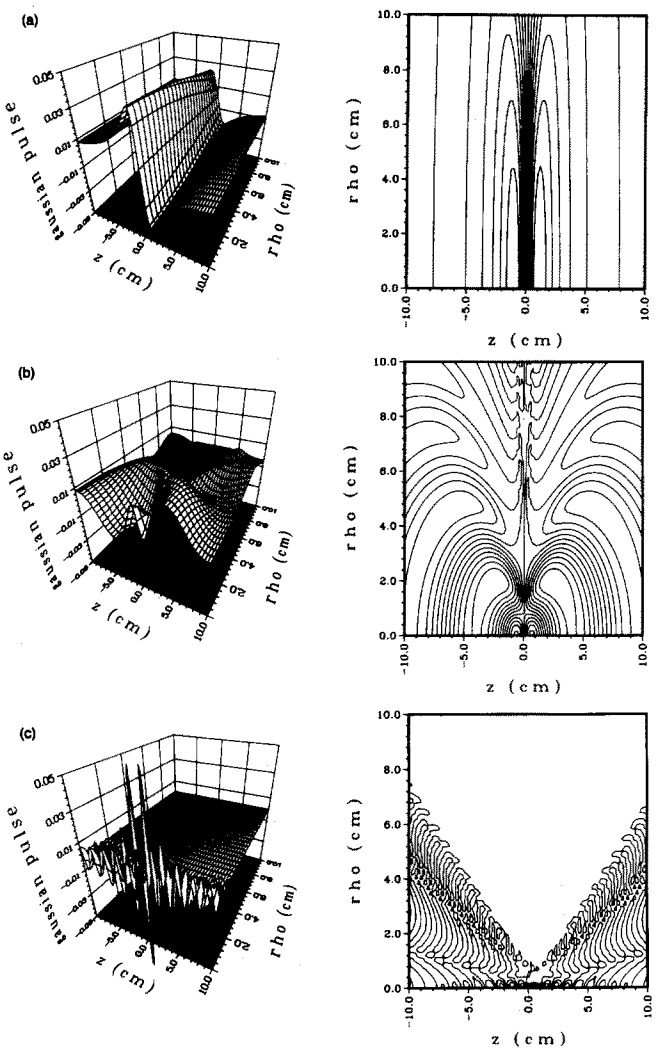

FIG. 2. As k increases, the Gaussian pulse profile becomes more concentrated along the z axis than along the ρ axis: (a) $f=3.0\times10^7$, (b) $f=3.0\times10^9$, (c) $f=3.0\times10^{11}$.

tion. It is the zeroth-order mode in a sequence of multipoles that can be generated in a cylindrical (rectangular) geometry by applying Laguerre (Hermite) polynomial operators to the fundamental Gaussian mode (7). Dr. Brittingham has brought to my attention that Bélanger[2] also recognized this point. However, contrary to the original article[1] and to Ref. 2, Fig. 1 demonstrates that the solution is neither focused nor packetlike nor a boost solution (translationally invariant). Moreover, recognizing that these pulses originate from complex source locations connects these results to a large body of literature. In particular, the concept that a Gaussian beam is *equivalent paraxially* to a spherical wave with a center at a (stationary) complex location was introduced by Deschamps[3] and was later used extensively by Felsen (for example, see Ref. 4) to model the propagation and scattering of Gaussian beams. In contrast to those beam descriptions, (7) is an *exact* solution of (1). On the other hand, the fundamental Gaussian pulse satisfies all of the properties associated with Gaussian beams. For example, its propagation through an optical system will be described by the $ABCD$ transformation law (see Ref. 5, Sec. 6.7).

The approach that led from (1) to (7) can also be used to define Gaussian pulse solutions of related equations of import. Consider the Klein–Gordon equation

$$\Box\Psi-\mu^2\Psi=0, \tag{8}$$

where $\mu=mc/\hbar$. It has the exact axially symmetric solution

$$\Psi(r,t)=\exp(-i\mu^2\tau/4k)\Phi(r,t)=\exp(i\mu^2ct\,2k)e^{ik_{\rm eff}\sigma}F_k, \tag{9}$$

where the effective modulation frequency

$$\omega_{\rm eff}=k_{\rm eff}c=(k-\mu^2/4k)c=[1-(mc/2\hbar k)^2]kc \tag{10}$$

has been modified by the mass of the particle. Note that this modulation frequency disappears when $\hbar k=mc/2$ leaving only the time harmonic portion $\exp(i\mu^2ct/2k)=\exp[i(mc^2)t/\hbar]$. Similarly, the wave equation in a transverse quadratic medium

$$\Box\Psi-(\epsilon_0+\epsilon_x x^2+\epsilon_y y^2)\Psi=0 \tag{11a}$$

reduces to a harmonic oscillator Schrödinger equation

$$4ik\,\partial_\tau F_k=-\Delta_\perp F_k+(\epsilon_0+\epsilon_x x^2+\epsilon_y y^2)F_k. \tag{11b}$$

Restricting the problem to one transverse spatial dimension, one can obtain an exact solution from path integral literature (e.g., see Ref. 6, Chap. 6) that is readily converted to one originating from a complex source location. A modified Gaussian pulse is obtained. In fact, when the transverse medium coefficients are small, the results reduce to those discussed above. One should then be able to modify standard quantum electronic results (e.g., see Ref. 5, Chap. 6) to apply directly to these exact complex center pulse solutions.

A strong objection to the results in Ref. 1 has been raised essentially because the solution (7) has infinite energy. This is not a drawback *per se*. Plane wave solutions of (1) also share the infinite energy property and are commonly employed in constructing physical signals. The Gaussian pulse solutions offer a new set of modes that can be used to construct finite energy solutions of (1). In particular, the function

$$\begin{aligned} f(\mathbf{r},t)\equiv h(\rho,\tau,\sigma)&=\int_0^\infty dk\,F(k)\,\phi_k(\rho,\tau,\sigma)\\ &=[4\pi i(z_0+i\tau)]^{-1}\int_0^\infty dk\,F(k)e^{-ks(\rho,\tau,\sigma)}, \end{aligned} \tag{12}$$

where $s(\rho,\tau,\sigma)=-i\sigma+\rho^2/(z_0+i\tau)$ satisfies (1) in real space-time. The wave number has been restricted to nonnegative values in this expansion (as well as assuming that $z_0>0$) to guarantee the finiteness of the kernel ϕ_k. The resulting Laplace transform expression yields a rich class of possible solutions. An inversion formula corresponding to the Gaussian pulse expansion (12) is

$$F(k)=\int_{-\infty}^{\infty}d\sigma\int_{-\infty}^{\infty}d\tau\int_0^\infty\rho\,d\rho\,\psi_k(\rho,\tau,\sigma)\,h(\rho,\tau,\sigma), \tag{13}$$

where the kernel is

$$\psi_k(\rho,\tau,\sigma)=8\pi^{1/2}e^{-(\tau/4kz_0)^2}\phi_k^*(\rho,\tau,\sigma), \tag{14}$$

ϕ_k^* being the complex conjugate of ϕ_k. Equivalently, the completeness relation

$$\int_{-\infty}^{\infty}d\sigma\int_{-\infty}^{\infty}d\tau\int_0^\infty\rho\,d\rho\,\psi_k(\rho,\tau,\sigma)\,\phi_{k'}^*(\rho,\tau,\sigma)=\delta(k-k') \tag{15}$$

is satisfied by ψ_k and ϕ_k. The density $\exp[-(\tau/4kz_0)^2]d\tau$ represents a Gaussian measure over τ with real variance $8(kz_0)^2$, kz_0 being the source phase distance, which guarantees the finiteness of the τ integration.

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Consider, as an example, the spectrum $F(k) = \exp(-ak)$. Equation (12) gives the pulse

$$f(\mathbf{r},t) = [4\pi i(z_0 + i\tau)]^{-1}\,[s(\rho,\tau,\sigma) + a]^{-1}. \tag{16}$$

Setting $f_+(\mathbf{r},t) = f(\mathbf{r},t)$, $f_-(\mathbf{r},t) = f_+(\mathbf{r},-t)$, and $a = z_0 = \gamma$, the composite pulse

$$\Psi(\mathbf{r},t) = f_+(\mathbf{r},t) - f_-(\mathbf{r},t) = \frac{1}{2\pi}\,\frac{\gamma(ct)}{\{[\rho^2 + (z-ct)(z+ct) + \gamma^2]^2 + 4\gamma^2(ct)^2\}} \tag{17}$$

is a real, exact solution of (1). A time sequence of a pulse (17) with $\gamma = 1.0$ is given in Fig. 3. The pulse has zero amplitude at $t = 0$ and its maxima occur at $\rho = z = 0$, for $0 < t \leqslant \gamma/c$ and lie on the sphere $\rho^2 + z^2 = R^2 = (ct)^2 - \gamma^2$, for $t > \gamma/c$. Its amplitude on that sphere $[8\pi\gamma(ct)]^{-1}$ decreases essentially as R^{-1} for $ct \gg \gamma$. The likeness of these figures to those describing a pebble dropped in a pond precipitated the name "splash pulse." As the figures illustrate, the support of the splash pulse is localized in space and separates space into two regions of null field for $t \gg \gamma/c$, the pulse layer being relatively thin. The apparent spikes in the surface subplot in Fig. 3(c) again are due to the coarseness of the computational grid employed in the graphics routine.

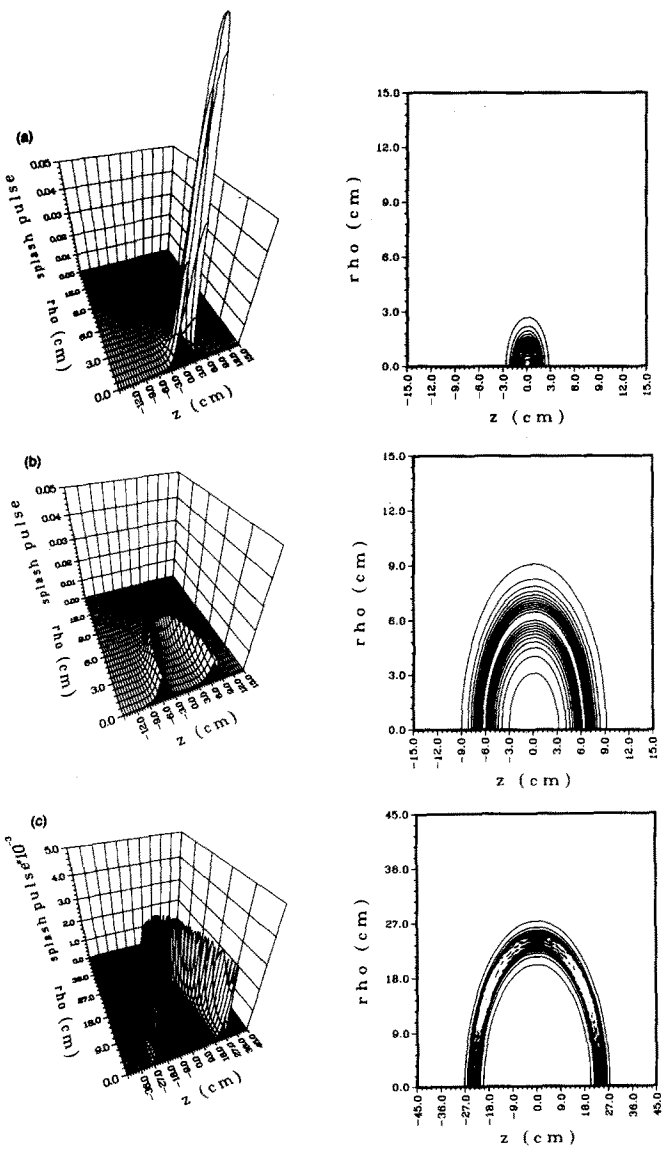

FIG. 3. Time sequence of the splash pulse (17) with $\gamma = 1.0$: (a) $t = 8.0\times10^{-11}$, (b) $t = 2.1\times10^{-10}$, (c) $t = 8.0\times10^{-10}$.

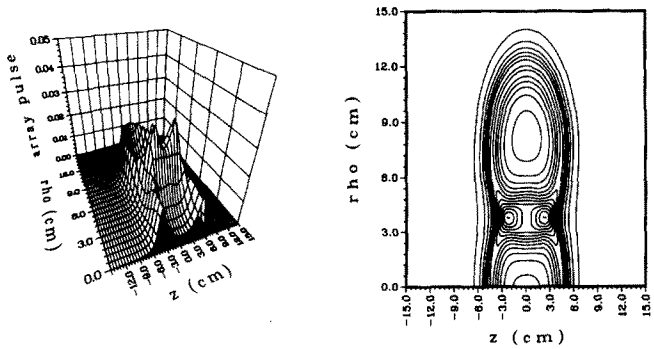

FIG. 4. The interaction of two splash pulses ($\gamma = 1.0$) confirms the linearity of the problem. The splash centers are $(\rho,z) = (0,0)$ and $(\rho,z) = (7.5,0)$.

The interaction of two splash pulses is depicted in Fig. 4. The apparent splash centers are the point $(\rho,z) = (0,0)$ and the ring $(\rho,z) = (7.5,0)$. The linear nature of the problem is reflected in the simple superposition in the overlap region and the decoupled propagation of each splash pulse.

III. CONCLUSION

Several issues remain outstanding and are currently under investigation. Foremost is the possible launchability of pulses derived from the fundamental Gaussian pulses. The physical connection between resonating structures and Gaussian beams (stationary complex center descriptions) leads one to speculate that such pulses may be associated with some special type of resonator cavity. In addition to the indicated k-superposition/transform pair, another class of solutions, those constructed by superposition of the complex source location z_0, may lead to other physically interesting pulses. Finally, with (2), nonlinear wave equations reduce immediately to the corresponding nonlinear Schrödinger equations. For instance, the cubic wave equation

$$\Box\Phi - \alpha|\Phi|^2\Phi = 0 \tag{18}$$

reduces to the cubic Schrödinger equation

$$4ik\,\partial_\tau F_k = -\Delta_\perp F_k + \alpha|F_k|^2 F_k. \tag{19}$$

At least for one transverse dimension, (19) has known soliton solutions (see Ref. 7, Sec. 5.3). Extensions of these solitons to ones associated with complex source locations may yield other physically interesting wave equation solutions.

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