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Diffraction of a nondispersive wave packet in the two slit interference experiment

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Abstract

In Young's two slit interference experiment, a single light particle or "photon" imparts its energy onto a dot on one of the light fringes of the diffracted field. In this work, it will be shown that a pulse solution of the scalar wave equation, or Maxwell's equations, going through a two slit screen reproduces the aforementioned single event interference effect. This solution, called the focus wave mode, is one of a class of solutions capable of accommodating both the wave and corpuscular aspects of light.

The formal interpretation of quantum mechanics, known as the Copenhagen interpretation [1], is based on probabilistic notions. Given an initial state, it can only give a probability for the system to exist in some final state. This probability is derived from a wave function which is a solution of a wave equation, e.g., the Schrödinger equation. The Copenhagen interpretation asserts that, as long as no measurements are carried out, no knowledge is possible concerning the intermediate states linking the final and the initial ones. Once a measurement is performed, the wave function collapses into one of the probable final states. Consequently, the concept of a trajectory disappears and complementary notions defining a trajectory like position and momentum cannot be specified exactly, but are limited by the uncertainty principle. In the same spirit, complementary concepts like those of particle and wave properties of matter are considered as two manifestations of reality that cannot be observed simultaneously, but can only reveal themselves in different situations. For example, the corpuscular nature of light is revealed in the photoelectric and Compton effects. On the other hand, the wave nature of light reveals itself in interference experiments, the first one of which was Young's two slit experiment.

In a typical setting for a two slit experiment, light is shone on a screen containing two narrow slits and is observed on a photographic plate placed behind the screen. The wave nature of light is directly observed since waves going through either slit will interfere constructively or destructively upon reaching the photographic plate, thus, forming interference fringes. The localized and compact nature of particles, on the other hand, makes

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it more difficult to explain the existence of the interference pattern. One way to circumvent such a difficulty is to assume that a large number of light particles or "photons" interact with each other, so that most of them end up on the light fringes. This was proved to be incorrect when very low intensity beams of light were used [2]. In such a case, single photons go through the screen one at a time and always end up on one of the light fringes. A single photon with transverse dimensions much less then the width of the slits, thus, senses both openings and produces interference effects. This raises the crucial question: how can a compact light particle, passing through one of the slits, feel the existence of the other one? Several interpretations of the quantum theory attempted to answer this question. The most accepted among these is the probabilistic Copenhagen interpretation [1]. Other interesting, but less accepted, interpretations were introduced, including the Broglie's "double solution" theory [3], Bohm's causal interpretation [4], which makes use of the notion of a "quantum potential", and the atemporal transactional interpretation [5] due to Cramer. These theories make use of a "pilot wave", a "quantum potential", or an "advanced wave" to guide the particle through the screen and onto one of the light fringes of the interference pattern.

Recently, there has been a rising interest in nondispersive wave packet solutions to linear equations, and their possible use to represent stable particles undergoing single events [6–8]. Whereas Barut [6] has attempted to formulate a quantum theory of single events based on a general representation of localized lump wave solutions, Hillion [7] and the current authors [8] have chosen to work with a specific class of solutions. Nevertheless, any wave packet that has strictly finite dimensions should fail in reproducing the interference pattern of Young's two slit experiment. Such a claim should be qualified by the reasonable assumption that the dimension of the wave packet is much smaller than the distance between the two slits. It is our aim in this paper to point out the existence of an exact pulse solution to Maxwell's equations that can accommodate both the wave and corpuscular aspects of light [7,8], and that such a pulse can produce single-event interference effects in the presence of a diffracting object. This pulse solution is known as the focus wave mode (FWM) [9]. In the sequel we shall consider only the scalar form of the FWM [10]. An extension to the vector-valued case follows immediately from the results of this paper.

The three-dimensional, zeroth order, FWM pulse

$$\Psi(\boldsymbol{r}, t) = \frac{A(\beta)}{4\pi[a_1 + \mathrm{i}(z-ct)]} \exp\left(-\frac{\beta\rho^2}{a_1 + \mathrm{i}(z-ct)}\right) \exp[\mathrm{i}(z+ct)] \tag{1}$$

is an exact solution to the scalar wave equation [10]. Here, z is the direction of propagation, t is the time, c is the speed of light, ρ is the transverse radial variable, β is a characteristic wave number and a_1 is a parameter determining the width of the pulse. The FWM solution does not disperse for all time, it travels in straight lines with velocity c, it is continuous and it is nonsingular for all points of space–time. The energy density of the field of the FWM, proportional to $|\Psi(\boldsymbol{r}, t)|^2$, has a Gaussian profile moving in the positive z-direction. At the center of the pulse ($z=ct$, $\rho=0$), the waist of the profile w is equal to $(a_1/\beta)^{1/2}$. For a_1 very small (i.e., for a pulse of a small waist), the solution given in (1) behaves like a localized Gaussian pulse moving in the positive z-direction [11], while for large a_1, such that $\beta a_1 \gg 1$, the solution looks more like a plane wave moving in the negative z-direction. Under the more stringent condition $\beta a_1 \ll 1$, the pulse has a large amplitude proportional to $1/a_1$ around its center, where $z-ct<a_1$ and $\rho<(a_1/\beta)^{1/2}$. Outside this portion of the pulse the amplitude falls off as $1/(z-ct)$ along the direction of propagation. At the same time the Gaussian envelope stretches out in the transverse direction, with a waist $(z-ct)/(\beta a_1)^{1/2}$ that increases as we move away from the central part of the field. Far from the center, i.e. for $|z-ct| \gg a_1$, the Gaussian envelope stretches out significantly such that its amplitude decreases and the energy density of that portion of the field becomes very small. Such a weak and extended field structure provides the localized central particle-like bump with its nonlocal wave-like attributes.

The FWM, with the appropriate choice of $a_1 \ll 1$ m, thus looks like a localized bump field with a large amplitude around the center, incorporated in an extended nonlocal field of very low amplitude. This pulse is only localized around $z=ct$ within the waist w, while the rest of its field fans out to cover relatively larger distances in the transverse direction away from the center. It is this property that allows the field of the FWM to feel the two slits in Young's experiment, even though the central bump field is fairly localized. For such a field one can associate the corpuscular aspects of light with its central portion of large amplitude. This localized bump is a part of an extended wave structure of much lower field intensities. Thus, the wave and particle aspects of light can be brought together into a single framework. This interpretation agrees with de Broglie's conception of wave–particle duality, whereby the wave and the particle aspects of reality should exist simultaneously. It also reflect Einstein's conception of a particle as a highly concentrated "bunch" field that remains localized and does not disperse in free motion.

In what follows the FWM pulse will be used as a light particle incident on a screen containing two narrow openings. To simplify our calculations, the openings in the screen will be represented by Gaussian slits of width $2b$, modelled by the following function,

$$S(y)=\frac{1}{2\sqrt{\pi}\,b}\left[\exp\left(-\frac{(y-d)^2}{2b^2}\right)+\exp\left(-\frac{(y+d)^2}{2b^2}\right)\right]. \tag{2}$$

The length of the slits is assumed to be much larger than their width; thus, the problem is basically two-dimensional. The screen is placed at $z=0$ along the transverse y-axis and the centers of the slits are situated at $y=\pm d$. The corresponding 2-D FWM can be derived from a slight modification of the spectrum of the 3-D FWM and by making use of the following synthesis [11],

$$\Psi_{\rm inc}(y,z,t)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dk_y\int_{-\infty}^{\infty}dk_z\int_{0}^{\infty}d\omega\,\Phi_{\rm inc}(k_y,k_z,\omega)\exp[-i(k_y y+k_z z)]\exp(i\omega t)\delta(\omega-c\sqrt{k_y^2+k_z^2}). \tag{3a}$$

The spectrum

$$\Phi_{\rm inc}(k_y,k_z,\omega)=\frac{\pi A(\beta)}{\sqrt{k_y^2+k_z^2}}\delta(\sqrt{k_y^2+k_z^2}-k_z-2\beta)\exp(ik_y y_0)\exp[-\tfrac{1}{2}(\sqrt{k_y^2+k_z^2}+k_z)a_1] \tag{3b}$$

gives the 2-D pulse

$$\Psi_{\rm inc}(y,z,t)=\frac{A(\beta)}{4\sqrt{\pi\beta[a_1+i(z-ct)]}}\exp\left(-\frac{\beta(y-y_0)^2}{a_1+i(z-ct)}\right)\exp[i(z+ct)]. \tag{3c}$$

where y_0 is the center of the incident pulse and we have assumed that the pulse and we have assumed that the pulse reaches the screen at time $t=0$. The spectrum $\Phi_{\rm out}(k_y,k_z,\omega)$ of the pulse beyond the screen is obtained by convolving the spectrum of the incident pulse (3b) with the transverse spectral distribution of the two slits resulting from the Fourier transformation of $S(y)$. The field $\Psi_{\rm out}(y,z,t)$ beyond the two slits can be obtained by substituting the spectrum $\Phi_{\rm out}(k_y,k_z,\omega)$ into a synthesis similar to that given in Eq. (3a). After carrying out the integrations over k_z and ω it follows that

$$\Psi_{\rm out}(y,z,t)=\frac{A(\beta)}{16\pi^2\sqrt{2\beta}}\int_{-\infty}^{\infty}dk_y\int_{-\infty}^{\infty}dk'_y\exp[-\tfrac{1}{2}b^2(k_y-k'_y)^2]\{\exp[id(k_y-k'_y)]+\exp[-id(k_y-k'_y]\}$$
$$\times\exp(-ik_y y)\exp(ik'_y y_0)\exp[-(k'^2_y/4\beta)a_1]\exp\{-iz(k'^2_y/4\beta-\beta)\}\exp\{ict\sqrt{(k'^2_y/4\beta-\beta)^2+k^2_y}\}. \tag{4}$$

The waist of the pulse is expected to be much less than the width of the slits; i.e., $w\ll 2b$. For a typical optical interference experiment we can have $2b=10^{-5}$ m, $2d=10^{-4}$ m and $\beta=10^7$ m^{-1}. The parameter a_1 can be chosen to be equal to 10^{-15} m. In this case the pulse width w equals 10^{-11} m, which is much smaller than $2b$. The integration given in Eq. (4) is very difficult to evaluate exactly because of the square root argument of the last exponential term in the integrand. A paraxial approximation is not possible since there are no restrictions on the values of k_y and k'_y. To overcome such a difficulty the change of variables

$$k_y=\tau\,,\qquad k'_y-\frac{2b^2}{2b^2+w^2}k_y=\sigma\,, \tag{5}$$

is utilized; it reduces Eq. (4) to the following form,

$$\begin{aligned}\Psi_{\rm out}(y,z,t)=&\frac{A(\beta)\exp(\mathrm{i}\beta z)}{16\pi^2\sqrt{2}\,\beta}\int\limits_{-\infty}^{\infty}\mathrm{d}\sigma\int\limits_{-\infty}^{\infty}\mathrm{d}\tau\exp(-\tfrac{1}{2}\sigma^2b^2)\exp(-\tfrac{1}{4}\tau^2w^2)\\&\times\{\exp[\mathrm{i}(dw^2/2b^2)\tau]\exp(-\mathrm{i}d\sigma)+\exp[-\mathrm{i}(dw^2/2b^2)\tau]\exp(\mathrm{i}d\sigma)\}\\&\times\exp[-\mathrm{i}\tau(y-y_0)]\exp(\mathrm{i}\sigma y_0)\exp[-\mathrm{i}z(\sigma^2+2\sigma\tau+\tau^2)/4\beta]\\&\times\exp[\mathrm{i}(ct/4\beta)\sqrt{(4\beta^2+\tau^2)^2-2\sigma^2(4\beta^2-3\tau^2)-4\sigma\tau(4\beta^2-\tau^2)+4\sigma^3\tau+\sigma^4}\,]\,.\end{aligned} \tag{6}$$

In this expression we have neglected w^2 compared to b^2. With $2b=10^{-5}$ m, the integration over σ has a bandwidth of 4×10^5 m^{-1}, so $\sigma<\beta=10^7$ for all values of σ contributing to the integral. For a photographic plate placed at $z=L$, our goal is to find the shape of the FWM pulse upon its arrival at $z=ct_L=L$. Using the property that $\sigma/\beta<1$, together with a binomial expansion of the square root, neglecting terms of order σ^4 and $\tau\sigma^3$ and carrying out the integration over σ, we obtain

$$\begin{aligned}\Psi_{\rm out}(y,L,t_L)=&\frac{A(\beta)\exp(2\mathrm{i}\beta L)}{16\pi^{3/2}\beta}\int\limits_{-\infty}^{\infty}\mathrm{d}\tau\frac{1}{\sqrt{\Delta}}\exp(-\tfrac{1}{4}\tau^2w^2)\exp[-\mathrm{i}\tau(y-y_0)]\\&\times(\exp\{-[d-y_0+2L\Lambda(\tau)]^2/2\Delta\}+\exp\{-[d+y_0-2L\Lambda(\tau)]^2/2\Delta\})\,,\end{aligned} \tag{7}$$

where terms of order $w^2\,d\tau/2b^2$ have been set equal to zero,

$$\Delta\equiv b^2+\mathrm{i}\frac{L}{B}\frac{1-\tau^2/4\beta}{1+\tau^2/4\beta}\,,\qquad \Lambda(\tau)\equiv\frac{\tau/2\beta}{1+\tau^2/4\beta}\,.$$

Although the integral in (7) seems to be quite difficult to evaluate, general features of the solution can be obtained. First, it should be noted that the integrand has a large bandwidth ($w^{-1}\simeq10^{11}$ m^{-1}) determined by the term $\exp(-\frac{1}{4}w^2\tau^2)$. The rest of the terms contribute to a depression in the amplitude of the integrand that decays to zero at $\tau=2\beta$. The center of the depression at 2β is much smaller than w^{-1}; as a consequence, most of the contributions to the integral come from the high frequency components, while the low frequency components contributions are relatively small and are only significant at the tails. We seek to estimate the order of the error introduced when the low frequency components are neglected and only the high frequency portion is retained, for which $\Delta\simeq\Delta'=b^2-\mathrm{i}(L/\beta)$ and $\Lambda(\tau)\simeq2\beta/\tau$. Pulling the term $1/\sqrt{\Delta'}$ out of the integration, the integrand has a maximum value of one. For $L\gg d$, e.g., $L=0.2$ m, the amplitude of the integrand is exponentially small for $\tau<2\beta$ and increases to a maximum of unity for $\tau>10\beta^2b\simeq5\times10^9$ m^{-1}. The amplitude stays at this value until it decays to zero for $\tau>4w^{-1}\simeq4\times10^{11}$ m^{-1}. Hence, neglecting the quantity $4L\beta/\tau$ compared to $d\pm y_0$ sets the amplitude for $\tau<5\times10^9$ m^{-1} equal to unity and introduces an error of order of magnitude less than 10^{-2} times the peak value of the pulse around its center. This factor contributes significantly only to the tails of the pulse. If such an approximation is adopted, the integration over τ can be easily carried out to give

$$\begin{aligned}\Psi_{\rm out}(y,L,t_L)=&\frac{A(\beta)\exp(2\mathrm{i}\beta L)}{8\pi\beta w}\exp[-(y-y_0)^2/w^2]\\&\times\frac{\exp[-(d^2+y_0^2)/2(b^2-\mathrm{i}L/\beta)]}{\sqrt{b^2-\mathrm{i}L/\beta}}\left[\exp\left(\frac{dy_0}{b^2-\mathrm{i}L/\beta}\right)+\exp\left(-\frac{dy_0}{b^2-\mathrm{i}L/\beta}\right)\right].\end{aligned} \tag{8}$$

It should be pointed out that what we really measure is the field's energy density which is proportional to $|\Psi_{out}|^2$. For $L^2/\beta^2 \gg b^4$, the square of the amplitude of the pulse is approximately given by

$$|\Psi_{out}(y, L, t_L)|^2 = \frac{A^2(\beta)}{16\pi^2 L^2 w^2} \exp[-2(y-y_0)^2/w^2] \exp[-b^2\beta^2(d^2+y_0^2)/L^2] \cos^2(\beta d y_0/L) . \quad (9)$$

This is the main result of this paper, where the field intensity is proportional to that of a Gaussian pulse of width w centered around $y = y_0$, multiplied by the interference term $\cos^2(dy_0\beta/L) \exp(-b^2\beta^2 y_0^2/L^2)$. For $\beta = 2\pi/\lambda$, the distance between two zeroes of the interference term is $\lambda L/2d$ which is the distance separating the fringes. As the pulse approaches the screen (for $z \gg ct$), its precursor extended field is symmetrically distributed over the two slits. This symmetry of the field with respect to the slits produces an interference effect. The resulting interference pattern, for a single pulse, will have such an extremely low energy density that it cannot be detected, except around its central bump. As the central portion of the pulse goes through the screen, it will be guided by its precursor extended field onto a point on the photographic plate. When the center of the pulse does not coincide with a zero of the interference pattern, the energy of the central portion will be delivered to an atom (or an electron in an atom) of the photosensitive emulsion. This will induce a reaction that will appear as a dot on the plate. Such a blackened dot occurs on one of the light fringes of the unfolding interference pattern. Since the field is much weaker at the tails, its energy will be too small to be detected. A light beam containing a large number of such wave packets or light particles would ultimately produce enough blackened dots to form the whole interference pattern. When the centers of the pulses end up on the zeroes of the interference pattern, the amplitudes of the pulses will be multiplied by zero and no atoms of the emulsion will be excited, resulting in dark fringes. Furthermore, it should be pointed out that just behind the screen, i.e., for $b^4 \gg L^2/\beta^2$, the square of the amplitude will be proportional to $\exp[-2(y-y_0)^2/w^2]$ multiplied by $\exp[-(d \pm y)^2/b^2]$. This will give exponentially small values for $y_0 \pm d > 2b$. Thus, detectors placed behind the screen will only detect photons going through one of the two slits.

For a large number of photons hitting the photographic plate simultaneously, tails that have been neglected in the case of a single photon might add up and disrupt the interference pattern. This is not the case, however, because not all the photons will have the same center y_0. So, we have to integrate the square of the amplitude of the pulse given in (9) over all possible values of y_0. For example, we can assume that the centers of the photons in the incident beam have a Gaussian distribution

$$\mathscr{D}(y_0) = \frac{N}{\sqrt{\pi}} \exp(-y_0^2/\Sigma^2) ,$$

where $\Sigma \gg w$ represents the width of the beam and N is a factor related to the number of photons. An integration over y_0 of the individual intensities $|\Psi_{out}(y, L, t_L)|^2$ in Eq. (9) multiplied by the distribution $\mathscr{D}(y_0)$ will result in a total square amplitude $|\Psi_{out}^{(T)}(y, L, t_L)|^2$ independent of the transverse variable y_0. Specifically, for typical values of the parameters involved one obtains the total field intensity

$$|\Psi_{out}^{(T)}(y, L, t_L)|^2 = \frac{A^2(\beta)N}{16\pi^2 L^2 w} \exp(-y^2/\Sigma^2) \exp[-b^2\beta^2(d^2+y^2)/L^2] \cos^2(\beta d y/L) . \quad (10)$$

The individual Gaussian pulse of width w centered around $y = y_0$ has disappeared, and the macroscopic interference term $\cos^2(\beta dy/L) \exp(-b^2\beta^2 y^2/L^2)$ has replaced the one dependent on the specific center y_0. One should notice the difference between the present calculation and an approach based upon an integration over the incident amplitudes $\Psi_{inc}(y, z, t)$. The analysis leading to Eq. (10) emphasizes the individualistic character of the wave packets used to represent the photons. Here the integration is over the intensities of such wave packets as they arrive at the photographic plate; i.e. we are summing up all the blackened dots on the photographic plate. On the other hand, an approach that integrates the fields of the incident wave packets produces a macroscopic incident field which is equivalent to the plane wave case. Thus, the model adopted in this paper can accommodate the results of two slit experiments involving single events (occurring sequentially or at the same time) for photons having different centers y_0.

In this paper, we have shown that a special nondispersive wave packet, namely the focus wave mode, can be used to explain how a single photon ends up on one of the light fringes in the presence of a screen containing two slits. Unlike de Broglie's singular solution [3], the FWM pulse is continuous and nonsingular with very large amplitudes around its center. This field, which does not have a finite total energy content, consists of a central "bump" of very large amplitudes sitting on an extended "background" field of very small amplitudes. The "background" field provides the FWM pulse with a nonlocality that should be contrasted with the localization of the central bump. The localized bump can be used to represent a particle while the rest of the field will behave as a wave.

An approach, that adopts nondispersive wave packets to represent stable particles undergoing single events, is not just a new interpretation of quantum mechanics, but an entirely new path that ought to be investigated [6–8]. Besides being capable of explaining the results of Young's experiment, wave packets such as the FWMs can account for the wave–particle duality. Also, the process dealt with in this paper is nonlocal because the field exists everywhere, but it is causal in the sense that the central particle-like portion of the field can be traced starting with its initial position going through the screen until it ends up on the photographic plate. The use of nondispersive wave packets in modelling particles [6–8,12] (e.g. photons) emphasizes the importance of their localization over the finiteness of their total energy content. Such wave packets can have very large amplitudes around their centers compared to their tails. Thus, one can assume that the energies usually observed in experiments result mainly from interactions of such central bumps. The waist of the central portion of the FWM pulse $w=\sqrt{a_1/\beta}$ decreases as the frequency $\omega=\beta c$ increases. Thus, a photon modelled by such a pulse becomes more localized as the frequency is increased. Since the energy of a photon increases with frequency, thus, the field of the central bump of the FWM pulse is expected to become more intense as it becomes more localized. It is worth mentioning that a similar behavior of the photon field has been suggested by Bacry [13] within a group-theoretic approach to the photon localization problem [14].

One should stress, however, that even though the wave packets we are dealing with have an infinite total energy content, finite energy structures could be constructed as a superposition of the FWM pulses around a mean wave number β_0. This could be done along the lines of Barut and Bracken's interesting attempt to construct finite energy solutions by superimposing spherical electromagnetic waves exhibiting particle behavior as they propagate in free-space [15]. In particular, such solutions have localized envelopes that travel with a group velocity $v_g<c$ and a phase velocity $v_{ph}>c$. In the aforementioned work [15], Barut and Bracken have been able to show that a superposition over a narrow spectrum with a mean frequency ω_0 produces a massive particle-like wave packet that has an energy proportional to ω_0 and a spin varying linearly with the angular momentum parameter m.

Finally, it should be emphasized that the wave function used here is a classical entity that travels in physical space, in sharp contrast with the quantum mechanical wave function that exists only in an abstract space. As a consequence, there is no collapse of the wave function, but instead we have the action of the central part of the field which can represent a light particle. Furthermore, the extended wave structure may provide a basis for the existence of a "quantum potential" [4]; or can be linked to the notion of an atemporal transaction [5]. The notions of a "pilot wave" or a "quantum potential" introduced to guide the particle through the screen are replaced in our work by the precursor field of the FWM pulse.

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