Localized energy pulse trains launched from an open, semi-infinite, circular waveguide

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A new decomposition of exact solutions to the scalar wave equation into bidirectional, backward and forward traveling plane waves is described. These elementary blocks constitute a natural basis for synthesizing Brittinghamlike solutions. Examples of such solutions, besides Brittingham's original modes, are Ziolkowski's electromagnetic directed energy pulse trains (EDEPTs) and Hillion's spinor modes. A common feature of these solutions is the incorporation of certain parameters that can be tuned in order to achieve slow energy decay patterns. The aforementioned decomposition is used first to solve an initial boundary-value problem involving an infinite waveguide. This is followed by considering a semi-infinite waveguide excited by a localized initial pulse whose shape is related directly to parameters similar to those arising in Ziolkowski's EDEPT solutions. The far fields outside the semi-infinite waveguide are computed using Kirchhoff's integral formula with a time-retarded Green's function. The resulting approximate solutions are causal, have finite energy, and exhibit a slow energy decay behavior.

I. INTRODUCTION

A few years ago, Brittingham[1] proposed a search for packetlike solutions to the homogeneous Maxwell's equations with the properties that they (1) are continuous and nonsingular, (2) have a three-dimensional pulse structure, (3) are nondispersive for all time, (4) move at the velocity of light in straight lines, and (5) carry finite electromagnetic energy. Such solutions have been termed focus wave modes (FWMs).

Brittingham was successful in proving that the FWMs satisfy the homogeneous Maxwell's equations together with the first four of the aforementioned properties. The original FWM integrals were found to be of infinite energy. To remedy that shortcoming, Brittingham introduced two infinitely extended surfaces of discontinuities that travel along the direction of propagation of the FWMs and divide space into three regions. The fields in the central region between these two surfaces were chosen to be equal to the original FWM integrals, while the fields outside were identically set equal to zero.

Difficulties arose immediately with the FWMs, especially with respect to their energy content. Wu and King[2] showed that Maxwell's equations cannot be satisfied across the discontinuities introduced by Brittingham. Consequently, it was established that the FWMs are characterized by infinite energy. That assertion was corroborated by the work of Sezginer,[3] followed by that of Wu and Lehman[4] who proved that any finite energy solution will involve the spreading of energy.

The work of Belanger,[5,6] Sezginer,[3] and Ziolkowski[7] showed that the original FWMs can be related to exact solutions of the three-dimensional scalar wave equation. Such solutions, which are expressed as products of a plane wave moving in the negative z direction with velocity c and an envelope function depending on x, y, and $z-ct$, will be termed the scalar FWMs in the sequel. All three authors indicated that the envelope function itself obeys exactly a complex parabolic equation, and demonstrated the intimate relation of the FWMs to the solutions arising from the paraxial approximation to the wave equation. Belanger[5] and Sezginer[3] showed that the FWMs can easily be written in terms of Gaussian–Laguerre and Gaussian–Hermite packetlike solutions. Later, Belanger[6] demonstrated that a Gaussian monochromatic beam can be observed as a Gaussian packetlike beam when the observer's inertial frame is moving at the same speed as the wave. Meanwhile, Ziolkowski[7] made the significant observation that the scalar FWMs describe fields that originate from moving complex sources. This observation linked the FWMs with earlier work by Deschamps[8] and Felsen[9] describing Gaussian beams as fields equivalent paraxially to spherical waves with centers at stationary complex locations.

At this stage, the main objection to the FWMs was their infinite energy content. It was pointed out by Ziolkowski,[7] however, that plane waves share with the FWMs their infinite energy property, and it was demonstrated that a superposition of these modes can produce finite energy. Ziolkowski also pointed out that since these modes are localized in space, a superposition of the FWMs might have an advantage over plane waves when it comes to describing the transfer of directed pulses in free space. Such pulses, characterized by high directionality and slow energy decay, were called electromagnetic directed energy pulse trains (EDEPTs), and it was argued by Ziolkowski[10] that they could be launched from a finite size antenna array.

EDEPTs share their high directionality and slow energy

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decay properties with the electromagnetic missile solutions introduced by Wu,[11] who argued that the electromagnetic energy density transmitted by a finite aperture under transient excitation does not have to decrease as R^{-2} when $R \to \infty$. The energy reaching the receiver has to decay eventually to zero. Wu demonstrated that one can make the product of the missile's cross-sectional area and the average energy per unit area approach zero as slowly as one wishes by choosing suitable frequency components of the exciting current. Wu deduced his results using the total received electromagnetic energy. Lee,[12] on the other hand, used the Mellin transform to derive asymptotic expressions for the **E** and **H** field components of an instantaneously excited missile field. Later, Lee[13] rederived the field components for a source with a finite excitation time. An interesting account of the launchability of electromagnetic missiles from a point source was given by Wu, King, and Shen[14] using a spherical dielectric lens. They found that such electromagnetic missiles can be classified into strong and weak ones according to the corresponding critical points of differentiable maps in two dimensions.

The unusual finite energy pulse solutions introduced by Wu[11] and Ziolkowski[7,10] seem to have been predicted on the basis of theoretical investigations that differ substantially from each other. In both cases, however, the directionality aspects of the solutions depend greatly on the appropriate choice of their spectral components. Similar ideas were contemplated by Durnin[15] when he introduced the diffracting-free "Bessel beams," and he was able to demonstrate that such beams have a larger depth compared to Gaussian beams, even if their central spots have the same radii. This behavior, which has been verified experimentally by Durnin, Miceli, and Eberly,[16] can be attributed mainly to differences in the energy distribution over identical apertures. Durnin's monochromatic beams are composed of different spatial spectral components. Similarly to the EDEPTs and the electromagnetic missiles, the depth of monochromatic beams can be controlled by varying only their spatial spectral content or changing their energy distribution over the aperture. On the other hand, both temporal and spatial spectral components are required in synthesizing highly directional time-limited pulses.

Another development along these lines is the use of Brittingham's modes by Hillion[17] to provide solutions to the homogeneous spinor wave equation. These solutions are called spinor focus wave modes. A weighted superposition of such modes results in finite energy pulses. Hillion studied in detail the particular case of Bessel weight functions. We mention, finally, that a similarity reduction technique utilizing the Lorentz invariance of the scalar wave equation has been used by Sockell[18] to generate novel classes of Brittinghamlike modes, and to provide a group-theoretic explanation for the existence of the scalar FWMs.

It is clear that the various attempts to study and synthesize highly directional pulses and beams aim at the same goal. This leads one to wonder whether there is a deeper underlying reality. It is our aim in this work to uniformize some of these attempts and to address some of the unanswered questions concerning the physical realizability of such wave solutions. One major concern is the limited understanding of the energy decay patterns of pulses. This is highly reflected in the current literature which is mainly concerned with the propagation of monochromatic signals, or modulated cw signals. Such wave solutions, with a very narrow bandwidth, tend to obscure some of the physical properties of the propagation of pulses with finite time durations. Unlike narrow bandwidth signals, such pulses have infinite bandwidths which make a concept like the "far field region" totally ambiguous. This situation led Ziolkowski[7] to suggest that a superposition of Brittingham's FWMs is more appropriate for the synthesis of highly directional pulses, and that the nonlocality of plane waves contradicts the spirit of composing highly localized wave solutions.

The Brittingham–Ziolkowski formalism has been a more radical approach than any other, mainly because it calls upon a superposition method that differs significantly from a regular superposition of sinusoidal plane waves. This might prove to be very hard to handle mathematically, but at the same time it provides a fresh procedure with which new ideas might be introduced into the problem of propagation of nonsinusoidal pulses, and can point out some physical implications that may be concealed by formal procedures, e.g., the Fourier synthesis. Moreover, EDEPTs contain certain parameters that can be adjusted to increase their slow energy decaying range. If these parameters could be related to physically meaningful quantities, a systematic procedure could be established to synthesize highly directional pulses that propagate in free space with very little spreading. In the spirit of this discussion, it seems worthwhile to pursue a better understanding of the Brittingham–Ziolkowski formalism and to study the physical realizability of highly directional wave solutions.

To achieve these goals, a novel approach to the synthesis of wave signals is used. This method was introduced by Besieris (Ref. 19) in order to understand the salient features of the Brittingham–Ziolkowski formalism. Its scope is broader, however, and encompasses classes of problems altogether different from wave propagation *in vacuo*. Within the framework of this new approach, exact solutions of the scalar wave equation are decomposed into bidirectional, backward and forward, plane waves traveling along a preferred direction z, viz., $\exp[-i\alpha(z-ct)]\exp[i\beta(z+ct)]$. These bilinear expressions can be elementary solutions to the Fourier-transformed (with respect to x and y) three-dimensional wave equation provided that a constraint relationship involving α, β, and the Fourier variables dual to x and y is satisfied. Such elementary blocks constitute a natural basis for synthesizing Brittinghamlike solutions, such as Ziolkowski's EDEPTs and splash pulses, Hillion's spinor modes, and the Ziolkowski–Belanger–Sezginer scalar FWMs.

In Sec. II, we shall provide a short introduction to the aforementioned new decomposition, together with a comparison to the well-established Fourier decomposition. For the sake of simplicity, the discussion will be restricted to the case of the three-dimensional scalar wave equation. It will be demonstrated, next, that all the known Brittinghamlike solutions can be reproduced by choosing fairly simple spectra for the novel synthesis, in contradistinction to the more complicated spectra that would have to be utilized in the case of a Fourier synthesis. Other choices of spectra can result in other types of solutions that can be of some value. In Sec. III, a specific demonstration will be given in connection with an infinitely long circular cylinder excited by a localized initial pulse whose size is related directly to parameters similar to those arising in the EDEPT solutions. The case of a semi-infinite waveguide excited by the same initial pulse will be considered in Sec. IV. The radiation field from the open end of the waveguide is computed using Kirchhoff's integral formula with a time-retarded Green's function. An approximate evaluation of Kirchhoff's integral gives solutions that are causal, have finite energy and exhibit an unusual decay behavior. Like the EDEPTs, these approximate solutions contain certain parameters that can be adjusted in order to control the shape of the pulses as they propagate in free space. These parameters are related to meaningful physical quantities, e.g., the shape of the initial pulse, the cross-sectional area of the waveguide and its cutoff frequencies. These aspects will be discussed in detail, together with the range of validity of the approximate solutions.

II. A NOVEL BILINEAR DECOMPOSITION

A new decomposition principle for partial differential equations will be introduced in this section. The discussion will be limited (cf. Ref. 20 for a more general exposition) to the scalar wave equation, viz.,

$$(\nabla^2 - \partial_t^2)\Psi(\mathbf{r},t) = 0 , \quad (1)$$

where ∇^2 is the 3D Laplacian, and the velocity of light is normalized to unity. In cylindrical coordinates,

$$\nabla^2 = \partial_\rho^2 + \rho^{-1}\partial_\rho + \rho^{-2}\partial_\phi^2 + \partial_z^2 .$$

The operator L in (1) can be divided into two parts, namely

$$L_1 = \partial_\rho^2 + \rho^{-1}\partial_\rho + \rho^{-2}\partial_\phi^2 + \partial_z^2 \quad (2a)$$

and

$$L_2 = -\partial_t^2 . \quad (2b)$$

The eigenfunctions of L_1 are $J_n(\kappa\rho)e^{\pm in\phi}e^{\pm ikz}$ and $N_n(\kappa\rho)e^{\pm in\phi}e^{\pm ikz}$, where $J_n(\kappa\rho)$ and $N_n(\kappa\rho)$ are Bessel functions of the first and second kind, respectively, and the eigenvalues equal $-(\kappa^2 + k^2)$. The operator L_2 has eigenfunctions $e^{\pm i\omega t}$ with eigenvalues ω^2. An elementary solution to equation (1) is given by

$$\Psi_e(\mathbf{r},t) = [A_n J_n(\kappa\rho) + B_n N_n(\kappa\rho)]e^{\pm in\phi}e^{i(kz \pm \omega t)} , \quad (3a)$$

with the constraint

$$\kappa^2 + k^2 - \omega^2 = 0 . \quad (3b)$$

Neglecting $N_n(\kappa\rho)$ because of its infinite value at $\rho = 0$, a solution to (1) can be expressed, in general, in terms of the superposition

$$\Psi(\mathbf{r},t) = \frac{1}{(2\pi)^2}\sum_{n=0}^{\infty}\int_0^\infty d\kappa \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} dk\, A_n(\omega,k,\kappa) \times \kappa J_n(\kappa\rho)e^{+in\phi}e^{i(kz-\omega t)}\delta(\omega^2 - \kappa^2 - k^2) , \quad (4)$$

which is the familiar Fourier synthesis.

To introduce the new decomposition, the following change of variables is used:

$$z + t = \eta , \quad z - t = \zeta . \quad (5)$$

The operator L then can be partitioned into two parts, viz.,

$$L_1 = \partial_\rho^2 + \rho^{-1}\partial_\rho + \rho^{-2}\partial_\phi^2 \quad (6a)$$

and

$$L_2 = 4\partial_{\zeta\eta}^2 . \quad (6b)$$

The eigenfunctions of L_1 are $J_n(\kappa\rho)e^{\pm in\phi}$ and $N_n(\kappa\rho)e^{\pm in\phi}$ and its eigenvalues equal $-\kappa^2$. The operator L_2 has eigenfunctions $e^{-i\alpha\zeta}e^{i\beta\eta}$ with eigenvalues $4\alpha\beta$. An elementary solution to (1) has the form

$$\Psi_e(\mathbf{r},t) \equiv \Psi_e(\rho,\zeta,\eta) = [C_n J_n(\kappa\rho) + D_n N_n(\kappa\rho)]e^{\pm in\phi}e^{-i\alpha\zeta}e^{i\beta\eta} , \quad (7a)$$

with the constraint

$$\alpha\beta = \kappa^2/4 . \quad (7b)$$

If $N_n(\kappa\rho)$ is neglected, a general solution to (1) can be written as

$$\Psi(\rho,\zeta,\eta) = \frac{1}{(2\pi)^2}\sum_{l=1,-1}\sum_{n=0}^{\infty}\int_0^\infty d\kappa \int_0^\infty d(l\alpha)\int_0^\infty d(l\beta) \times C_n(l\alpha,l\beta,\kappa)\kappa J_n(\kappa\rho)e^{\pm in\phi} \times e^{-il\alpha\zeta}e^{il\beta\eta}\delta(\alpha\beta - \kappa^2/4) . \quad (8)$$

It should be noted that the limits of α and β are chosen to range from zero to infinity because the constraint (7b) restricts α and β to the same sign. As a consequence, the parameter l takes the values of -1 or $+1$.

The two representations (4) and (8) may not look the same, but there is a one-to-one correspondence between them through the change of variables

$$k = \beta - \alpha , \quad \omega = \beta + \alpha . \quad (9)$$

Any distinct advantages of these representations depends on the kind of solutions they represent. For certain types of solutions, (4) might be more suitable than (8), and vice-versa. The situation is more like choosing between cylindrical and Cartesian coordinates. For some problems it is more appropriate to use Cartesian coordinates, while for others cylindrical coordinates can be more suitable.

One class of problems for which the representation (8) can be very advantageous deals with Brittinghamlike solutions. Limiting the discussion to cases where $n = 0$, the following choices of $C_0(\alpha,\beta,\kappa)$ result in a number of known solutions of this type:

(i) The singular spectrum

$$C_0(\alpha,\beta,\kappa) = (\pi/2)\delta(\beta - \beta')e^{-\alpha a_1} \quad (10)$$

gives the scalar FWMs introduced by Belanger,[5] Sezginer,[3] and Ziolkowski[7]:

$$\Psi(\rho,\zeta,\eta) = \frac{1}{4\pi(a_1 + i\zeta)}\exp\left[-\beta'\left(\frac{\rho^2}{(a_1 + i\zeta)} - i\eta\right)\right] . \quad (11)$$

(ii) The spectrum

$$C_0(\alpha,\beta,\kappa) = (\pi/2)e^{-(\alpha a_1 + \beta a_2)} \quad (12)$$

yields Ziolkowski's[7] "splash pulse"

$$\Psi(\rho,\zeta,\eta)=\{4\pi[(a_2-i\eta)(a_1+i\zeta)+\rho^2]\}^{-1}. \quad (13)$$

(iii) The shifted spectrum

$$C_0(\alpha,\beta,\kappa)=\frac{\pi p}{2\Gamma(q)}(p\beta-b)^{q-1}e^{-[\alpha a_1+(p\beta-b)a_2]}, \quad \beta>b/p,$$
$$=0, \quad b/p>\beta\geqslant 0, \quad (14)$$

results in Ziolkowski's modified power spectrum (MPS) pulse[10]

$$\Psi(\rho,\zeta,\eta)=e^{-bs/p}/[4\pi(a_1+i\zeta)(a_2+s/p)^q], \quad (15)$$

where

$$s=\rho^2/(a_1+i\zeta)-i\eta.$$

(iv) The Bessel spectrum

$$C_0(\alpha,\beta,\kappa)=(\pi/2)J_\nu(\beta b)e^{-\alpha a_1} \quad (16)$$

gives rise to the wave function

$$\Psi(\rho,\zeta,\eta)=\frac{1}{\pi(a_1+i\zeta)}\,\frac{b^{-\nu}(\sqrt{s^2+b^2}-s)^\nu}{\sqrt{s^2+b^2}}, \quad (17)$$

which is the scalar analog of Hillion's solution to the spinor wave equation.[17]

Other types of solutions can be synthesized by choosing different spectra. An interesting spectrum is the following:

$$C_0(\alpha,\beta,\kappa)=(\alpha+\beta)^N e^{-(\alpha a_1+\beta a_2)}. \quad (18)$$

The corresponding wave function can be written out explicitly as

$$\Psi(\rho,\zeta,\eta)=\frac{1}{\pi}\sum_{n=0}^{N}\Gamma(N+1)\frac{F\{n+1,N-n+1;1;[-\rho^2/(a_1+i\zeta)(a_2-i\eta)]\}}{(a_2-i\eta)^{n+1}(a_1+i\zeta)^{N-n+1}}, \quad (19)$$

where $F(\alpha,\beta;\gamma;\delta)$ is the hypergeometric function.

The solutions cited above are of great interest because they can be designed to have a slow energy decay as they propagate in free space. Due to the simplicity of the corresponding spectra, the representation (8) provides the most natural basis for the synthesis of these classes of solutions (see Ref. 20 for more details). This aspect can be clarified further by using (9) to transform (8) into a Fourier representation. The corresponding Fourier spectra will prove to be very complex compared to those given above.

It is of interest to mention that the representation (8) has an inversion formula analogous to the Fourier one. If the integration over α in (8) is carried out with $l=1$, $\Psi(\rho,\zeta,\eta)$ assumes the following form:

$$\Psi(\rho,\zeta,\eta)=\frac{1}{(2\pi)^2}\int_0^\infty d\beta\int_0^\infty d\kappa\,\frac{1}{\beta}\,C_0\left(\frac{\kappa^2}{4\beta},\beta,\kappa\right)\kappa J_0(\kappa\rho)\times e^{-i(\kappa^2/4\beta)\zeta}e^{i\beta\eta}. \quad (20)$$

The inversion formula, which incorporates an exponential measure, is given explicitly as follows:

$$C_0\left(\frac{\kappa^2}{4\beta},\beta,\kappa\right)=\frac{\sqrt{\pi}}{2}\int_{-\infty}^{+\infty}d\zeta\,e^{-\zeta^2/16\beta^2}\int_{-\infty}^{+\infty}d\eta\int_0^\infty d\rho\times\rho J_0(\kappa\rho)\Psi(\rho,\zeta,\eta)e^{i(\kappa^2/4\beta)\zeta}e^{-i\beta\eta}. \quad (21)$$

It is straightforward to check its validity by using the explicit solutions $\Psi(\rho,\zeta,\eta)$ given earlier and checking whether one can derive the corresponding spectra.

III. THE INFINITE WAVEGUIDE

It will be demonstrated in this section that the representation (8) can be used to solve an initial boundary-value problem. A specific problem will be solved involving an infinite cylindrical waveguide. Towards this goal, consider the three-dimensional wave Eq. (1), viz.,

$$(\nabla^2-\partial_t^2)\Psi(\mathbf{r},t)=0,$$

with the initial conditions

$$\Psi(\mathbf{r},0)=F(\rho,z), \quad (22a)$$

$$\Psi_t(\mathbf{r},0)=G(\rho,z), \quad (22b)$$

and the boundary condition

$$\Psi(R,z,t)=0, \quad (22c)$$

where $F(\rho,z)$ and $G(\rho,z)$ are real functions and R is the radius of the cross section of an infinitely long circular cylinder.

In analogy to the free space solution (8), one can write directly the corresponding superposition for the bounded problem under consideration by replacing the integration over κ in (8) by a summation over the discrete values κ_{0m}; specifically,

$$\Psi(\rho,\zeta,\eta)=\frac{1}{(2\pi)^2}\sum_{m=1}^{\infty}\int_0^\infty d\alpha\int_0^\infty d\beta\,C_0(\alpha,\beta,\kappa_{0m})\times J_0\left(\frac{\kappa_{0m}\rho}{R}\right)e^{-i\alpha\zeta}e^{i\beta\eta}\delta\left(\alpha\beta-\frac{\kappa_{0m}^2}{4R^2}\right), \quad (23)$$

where κ_{0m} are the zeros of the zero-order Bessel function. It should be noted that the positive branch of α and β has been chosen; furthermore, without any loss of generality $\Psi(\rho,\zeta,\eta)$ has been assumed to be azimuthally symmetric (i.e., $n=0$).

A choice of spectrum analogous to that related to Ziolkowski's "splash pulse," namely,

$$C_0(\alpha,\beta,\kappa_{0m})=(\pi/2)e^{-(\alpha a_1+\beta a_2)},$$

leads to the solution

$$\Psi(\mathbf{r},t)=\mathrm{Re}\sum_{m=1}^{\infty}\left[\frac{1}{4\pi}J_0\left(\frac{\kappa_{0m}\rho}{R}\right)\times K_0\left(\frac{\kappa_{0m}}{R}\sqrt{(a_1+i\zeta)(a_2-i\eta)}\right)\right], \quad (24)$$

where K_0 is the zero-order modified Bessel function of the second kind. Our subsequent discussion will be restricted to a single mode with no loss of generality.

The solution (24), for a characteristic modal number m, is of some interest because it can model a fairly localized pulse propagating in a waveguide, where at $t=0$,

$$\Psi(\mathbf{r},0)=\frac{1}{4\pi}J_0\left(\frac{\kappa_{0m}\rho}{R}\right)\times\mathrm{Re}\left[K_0\left(\frac{\kappa_{0m}}{R}\sqrt{(a_1+iz)(a_2-iz)}\right)\right] \quad (25a)$$

and

$$\Psi_t(\mathbf{r},0)=\frac{\kappa_{0m}}{8\pi R}J_0\left(\frac{\kappa_{0m}\rho}{R}\right)\times\mathrm{Re}\left[iK_1\left(\frac{\kappa_{0m}}{R}\sqrt{(a_1+iz)(a_2-iz)}\right)\times\frac{a_1+a_2}{\sqrt{(a_1+iz)(a_2-iz)}}\right]. \quad (25b)$$

The initial pulse can be extremely localized for large a_2. This can be demonstrated by using the asymptotic expansion of K_0 for large arguments,[21] viz.,

$$K_0(z)\simeq\sqrt{\frac{\pi}{2z}}\,e^{-z}\left(1-\frac{1}{8z}+\frac{9}{2(8z)^2}+\cdots\right), \quad (26)$$

in order to rewrite the initial condition (25a) approximately as follows:

$$\Psi(\rho,z,0)\simeq\frac{1}{4\pi}J_0\left(\frac{\kappa_{0m}\rho}{R}\right)\sqrt{\frac{\pi R}{2\kappa_{0m}\sqrt{a_2z}}}\,e^{-\kappa_{0m}\sqrt{a_2z/2}/R}\times\cos\left(\frac{\kappa_{0m}}{R}\sqrt{\frac{a_2z}{2}}+\frac{\pi}{8}\right). \quad (27)$$

The initial pulse $\Psi(\mathbf{r},0)$ falls off exponentially along the z direction as $\exp(-\kappa_{0m}\sqrt{a_2z/2}/R)$. This approximate form is valid away from the pulse's center. To determine the shape of the initial pulse around $z=0$, one can use the small argument approximation of the modified Bessel function K_0, namely,[21]

$$K_0(z)\simeq-\ln(z)\,. \quad (28)$$

At $z=0$, the amplitude of the initial pulse $\Psi(\mathbf{r},0)$ has the value

$$\Psi(\rho,0,0)=\frac{1}{4\pi}K_0\left(\frac{\kappa_{0m}}{R}\sqrt{a_1a_2}\right)J_0\left(\frac{\kappa_{0m}\rho}{R}\right). \quad (29)$$

If the product of a_1a_2 is very small so that $\kappa_{0m}\sqrt{a_1a_2}/R\ll1$, Eq. (28) yields

$$\Psi(\rho,0,0)\simeq-\frac{1}{4\pi}\ln\left(\frac{\kappa_{0m}}{R}\sqrt{a_1a_2}\right)J_0\left(\frac{\kappa_{0m}\rho}{R}\right).$$

This expression shows that small values of $\sqrt{a_1a_2}$ correspond to large amplitudes of the initial pulse. Noting, also, that the oscillatory term $\cos[\pi/8+(\kappa_{0m}/R)\sqrt{a_2z/2}]$ in Eq. (27) depends on a_2, it is clear that the parameters a_1 and a_2 can now be related to physically meaningful quantities, such as the width of the initial pulse and its amplitude. A similar discussion applies to the second initial condition (25b) but will not be carried out here.

The behavior of the pulse solution (24) is very sensitive to the values of the parameters a_1 and a_2. If these parameters are equal, the solution (24) models a pulse that splits into two halves traveling in opposite directions. This situation is not optimal for the transmission of energy in the waveguide. A pulse traveling primarily in the positive z direction requires that $a_2>1$ and $a_1\ll1$, with the product $\kappa_{0m}\sqrt{a_1a_2}/R\ll1$. To see this, one can look at the pulse centers at $z=t$ and $z=-t$, where

$$\Psi(\rho,z,z)=\mathrm{Re}\left[\frac{1}{4\pi}J_0\left(\frac{\kappa_{0m}\rho}{R}\right)K_0\left(\frac{\kappa_{0m}}{R}\sqrt{a_1(a_2-i2z)}\right)\right] \quad (30)$$

and

$$\Psi(\rho,z,-z)=\mathrm{Re}\left[\frac{1}{4\pi}J_0\left(\frac{\kappa_{0m}\rho}{R}\right)K_0\left(\frac{\kappa_{0m}}{R}\sqrt{a_2(a_1+i2z)}\right)\right]. \quad (31)$$

As long as $2z\ll a_2$, $\Psi(\rho,z,z)$ in Eq. (30) remains constant with

$$\Psi(\rho,z,z)\simeq\frac{1}{4\pi}J_0\left(\frac{\kappa_{0m}\rho}{R}\right)K_0\left(\frac{\kappa_{0m}}{R}\sqrt{a_1a_2}\right). \quad (32)$$

This expression is identical to the amplitude of the initial pulse. We conclude that the parameter a_2 determines the range through which the pulse can travel before it starts decaying. Beyond this range the pulse starts decaying logarithmically, viz.,

$$\Psi(\rho,z,z)\simeq\frac{-1}{4\pi}J_0\left(\frac{\kappa_{0m}\rho}{R}\right)\ln\left(\frac{\kappa_{0m}}{R}\sqrt{2a_1z}\right), \quad (33)$$

as long as $\kappa_{0m}\sqrt{2a_1z}/R\ll1$. Thus, the pulse traveling in the positive z direction stays almost unchanged as long as $2z\ll a_2$, decays logarithmically for $\kappa_{0m}\sqrt{2a_1z}/R\ll1$, beyond which it dies off exponentially as $\exp(-\kappa_{0m}\sqrt{2a_1z}/R)$. As for the pulse traveling in the negative z direction, it can be easily seen that for $a_1\ll1$ the expression (31) can be simplified as follows:

$$\Psi(\rho,z,-z)=\mathrm{Re}\left[\frac{1}{4\pi}J_0\left(\frac{\kappa_{0m}\rho}{R}\right)K_0\left(\frac{\kappa_{0m}}{R}\sqrt{i2a_2z}\right)\right].$$

But for $a_2>10$ the center of the pulse in (31) will decay exponentially as $\exp(-\kappa_{0m}\sqrt{2a_2z}/R)$ and the pulse will die off within a very short distance from the origin.

With an appropriate choice of the parameters a_1 and a_2, one is capable of launching a pulse in one direction down an infinite cylinder. Another interesting aspect of this solution is its ability to resist the dispersive effects of the waveguide. It has been demonstrated that the pulse can travel with almost no decay in its center for a distance $\simeq a_2/2$, beyond which the decay in the center of the pulse is only logarithmic for $a_2/2<z\ll(R^2/2\kappa_{0m}^2a_1)$. For example, if one chooses $a_1=10^{-13}$ m, $a_2=10^3$ m and $\kappa_{0m}/R=1000\ \mathrm{m}^{-1}$, then the pulse can travel down the waveguide a distance of approximately 500 m without any decay in the pulse center, and will spread very little for the following 5×10^4 m. To see how little it does decay, consider the amplitude of the center of the initial pulse, which is proportional to $K_0(\kappa_{0m}\sqrt{a_1a_2}/R)$ or, asymptotically, to $-\ln(10^{-2})\simeq2\ln(10)$. The pulse

center after traveling a distance of 50 km will have an amplitude proportional to ln(10). Thus, over a distance of 50 km, the pulse's center decays only to half its original value.

One would hope that such pulses could still hold themselves together when launched in free space from an open end of the waveguide, and that they could still exhibit a slow decay behavior. In this case the pulses would be analogous to Ziolkowski's EDEPTs and the open waveguide could act as their source. This scheme will be pursued in the next section where it will be shown that a semi-infinite waveguide can be used as a source for slowly decaying pulses.

IV. THE SEMI-INFINITE WAVEGUIDE

A natural extension of the previous section is to consider the case of a semi-infinite waveguide. The initial pulse is the same as that given by Eq. (24) and the waveguide is open at the position $z = L$, where L is large enough so that the amplitude of the tail of the initial pulse is extremely small, and for all practical purposes can be taken equal to zero, while the waveguide appears to be practically infinite. It is assumed, furthermore, that the radius of the cylinder is large compared to the wavelengths of the excited modes. In this case, upon reaching the guide's opening, the pulse is basically launched in the forward direction and the reflected field from the edges of the cylinder can be neglected.

Under the assumptions made above, one can use Kirchhoff's diffraction formula to find an expression for the far field in terms of the field illuminating the aperture or the cylinder's opening. Kirchhoff's formula has the form[22]

$$\hat{\Psi}(x',y',z',\omega) = \frac{1}{4\pi}\int_S da\left[\frac{\partial}{\partial z}\hat{\Psi}(x,y,z,\omega)\frac{e^{-i\omega r}}{r} - \frac{\partial}{\partial z}\left(\frac{e^{-i\omega r}}{r}\right)\hat{\Psi}(x,y,z,\omega)\right], \qquad (34)$$

where (x',y',z') is the point of observation, the integration is carried over the area of the aperture, and the normal to the aperture is in the z direction. $\hat{\Psi}(x,y,z,\omega)$ is the Fourier transform of $\Psi(x,y,z,t)$ with respect to time, and is defined as follows:

$$\hat{\Psi}(x,y,z,\omega) = \int_{-\infty}^{+\infty} dt\, \Psi(x,y,z,t)e^{-i\omega t}. \qquad (35)$$

The variable r appearing in (34) is the distance between the aperture and the observation point; it is given explicitly as follows:

$$r = \sqrt{\rho^2 + \rho'^2 + 2\rho\rho'\cos(\theta - \theta') + (z' - L)^2}. \qquad (36)$$

Only the amplitude of the pulse along the line of sight ($\rho' = 0$) will be considered in the sequel. In this case, Kirchhoff's formula can be rewritten as

$$\hat{\Psi}(0,0,z',\omega) = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^R d\rho\,\rho\left[\frac{\partial}{\partial z}\hat{\Psi}(x,y,z,\omega)\times\frac{e^{-i\omega r}}{r} - \frac{\partial}{\partial z}\left(\frac{e^{-i\omega r}}{r}\right)\hat{\Psi}(x,y,z,\omega)\right]_{z=L}, \qquad (37)$$

with

$$r = \sqrt{\rho^2 + (z' - z)^2}\,\Big|_{z=L}. \qquad (38)$$

From Kirchhoff's integrals (34) and (37), together with the definition of the Fourier transform (35), it can be seen that the time-retarded Green's function has been used. As a consequence, the resulting pulse is causal and it propagates away from the aperture into free space.

As mentioned earlier, the initial pulse given in (24) will be used, with L large enough so that the waveguide appears to be infinitely long. For all practical purposes, the basic solution derived in Sec. III can be used to illuminate the open end of the waveguide. This solution is rewritten here for convenience, viz.,

$$\Psi(\mathbf{r},t) = \frac{1}{4\pi} J_0\left(\frac{\kappa_{0m}\rho}{R}\right)K_0\left(\frac{\kappa_{0m}}{R}\sqrt{(a_1 + i\zeta)(a_2 - i\eta)}\right). \qquad (39)$$

This pulse is located initially at the origin and with an appropriate choice of a_1 and a_2 it can be made to travel in the positive z direction. All the discussion pertaining to the initial conditions and the conditions required for localization are the same as in Sec. III.

Substituting (39) into Eqs. (34) and (37), Kirchhoff's integral formula (37) assumes the following form after an integration over ϕ:

$$\hat{\Psi}(0,0,z',\omega) = \frac{f_2(\omega)}{8\pi}\int_0^R d\rho\,\rho\frac{e^{-i\omega r}}{r}J_0\left(\frac{\kappa_{0m}\rho}{R}\right) - \frac{f_1(\omega)}{8\pi}\int_0^R d\rho\,\rho\left(\frac{(z'-L)}{r^3} + \frac{i\omega(z'-L)}{r^2}\right)e^{-i\omega r}. \qquad (40)$$

Here, $f_1(\omega)$ and $f_2(\omega)$ are defined as follows:

$$f_1(\omega) = \int_{-\infty}^{+\infty} dt\, e^{-i\omega t}K_0\left(\frac{\kappa_{0m}}{R}\sqrt{[a_1 + i(z-t)][a_2 - i(z+t)]}\right)\Big|_{z=L}, \qquad (41)$$

$$f_2(\omega) = \int_{-\infty}^{+\infty} dt\, e^{-i\omega t}\frac{\partial}{\partial z}K_0\left(\frac{\kappa_{0m}}{R}\sqrt{[a_1 + i(z-t)][a_2 - i(z+t)]}\right)\Big|_{z=L}. \qquad (42)$$

For $(z' - L) \gg \rho$, the quantity r can be approximated by

$$r \simeq (z' - L) + \rho^2/2(z' - L), \qquad (43)$$

and Eq. (40) can be rewritten as

$$\hat{\Psi}(0,0,z',\omega) = e^{-i\omega\hat{z}}\left[\frac{f_2(\omega)}{8\pi\hat{z}} - \frac{f_1(\omega)}{8\pi}\left(\frac{1}{\hat{z}^2} + \frac{i\omega}{\hat{z}}\right)\right]\int_0^R d\rho\,\rho\, J_0\left(\frac{\kappa_{0m}\rho}{R}\right)e^{-i\omega\rho^2/2\hat{z}}, \qquad (44)$$

where $\hat{z} = z' - L$. An evaluation of the integral

$$I = \int_0^R d\rho\, \rho\, J_0\left(\frac{\kappa_{0m}\rho}{R}\right) e^{-i\omega\rho^2/2\hat{z}} \tag{45}$$

is needed. This integral can be related to Lommel's function of two variables. Making use of the relations (3) and (4) on page 543 of Watson,[23] it is easy to show that

$$I = \frac{\hat{z}}{\omega} e^{-i\omega R^2/2\hat{z}} \left[-U_1\left(\frac{\hat{z}\kappa_{0m}^2}{\omega R^2}, \kappa_{0m}\right) + iU_0\left(\frac{\hat{z}\kappa_{0m}^2}{\omega R^2}, \kappa_{0m}\right)\right] + \frac{\hat{z}}{\omega}\left[U_1\left(\frac{\hat{z}\kappa_{0m}^2}{\omega R^2}, 0\right) - iU_0\left(\frac{\hat{z}\kappa_{0m}^2}{\omega R^2}, 0\right)\right]. \tag{46}$$

At this stage, one can go back to Eq. (44), find $\hat{\Psi}(0,0,z',\omega)$, carry out the inverse transform, and find $\Psi(0,0,z',t)$. The integrations become quite involved, however, and it will be a very tedious task to find a closed form solution. To simplify the integrals involved in the Fourier inversion, one can use the asymptotic expansion of $U_\nu(\mu,\lambda)$ for large μ, while ν and λ are kept constant. The quantity $\hat{z}\kappa_{0m}^2/\omega R^2$ entering into Eq. (46) must be large. A reasonable limit can be chosen as

$$\hat{z}\kappa_{0m}^2/\omega R^2 > 1000. \tag{47}$$

One can use the asymptotic expansions given in Eqs. (3) and (4) on page 550 of Watson[23] to write

$$U_\nu(\mu,0) \simeq \cos(\mu/2 - \nu\pi/2) + \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(\nu - 1 - 2m)(\mu/2)^{2m-\nu+2}} \tag{48}$$

and

$$U_\nu(\mu,\lambda) \simeq \cos\left(\frac{\mu}{2} + \frac{\lambda^2}{2\mu} - \frac{\nu\pi}{2}\right) + \sum_{m=0}^{\infty} (-1)^m \left(\frac{\lambda}{\mu}\right)^{2m-\nu+2} J_{\nu-2-2m}(\lambda). \tag{49}$$

Using these expansions in Eq. (46), it follows that

$$I = \frac{\hat{z}}{\omega} e^{-i\omega R^2/2\hat{z}} \sum_{m=0}^{\infty} (-1)^m \times \left[-\left(\frac{\omega R^2}{\hat{z}\kappa_{0m}}\right)^{2m+1} J_{1-2m}(\kappa_{0m}) + i\left(\frac{\omega R^2}{\hat{z}\kappa_{0m}}\right)^{2m+2} J_{-2-2m}(\kappa_{0m})\right].$$

For small values of $\omega R^2/\hat{z}\kappa_{0m}$ (e.g., <0.1), the first term in the above expression can be retained, while the rest of the terms are neglected. Thus, I can be approximated by

$$I = J_1(\kappa_{0m})(R^2/\kappa_{0m}) e^{-i\omega R^2/2\hat{z}} + O(1/\hat{z}). \tag{50}$$

This approximate result is valid for large $\hat{z}$ or, more generally, for large $\hat{z}\kappa_{0m}/\omega R^2$. This requirement follows from the condition (47) as long as $\kappa_{0m} < 30$, i.e., for the lowest ten modes. As a result, the approximate expression (50) is valid for the following value of $\hat{z}$:

$$\hat{z} > 1000(\omega R^2/\kappa_{0m}^2).$$

Since $\omega > \kappa_{0m}/R$ for the waveguide, the above inequality can be rewritten as

$$\hat{z} > 1000(R/\kappa_{0m}). \tag{51}$$

Substituting (50) into (44), the wave function $\hat{\Psi}(0,0,\hat{z},\omega)$ along the line of sight can be found to be equal to

$$\hat{\Psi}(0,0,\hat{z},\omega) = \frac{J_1(\kappa_{0m})R^2}{8\pi\kappa_{0m}\hat{z}} \exp\left[-i\omega\left(\hat{z} + \frac{R^2}{2\hat{z}}\right)\right] \times [f_2(\omega) - i\omega f_1(\omega)], \tag{52}$$

where all terms proportional to $\hat{z}^{-2}$ have been neglected. This approximate result is quite accurate for the range determined by (51). It is not hard, now, to find the inverse Fourier transform of the expression (52); it can be expressed as

$$\Psi(0,0,\hat{z},t) = \frac{J_1(\kappa_{0m})R^2}{8\pi\kappa_{0m}\hat{z}} [F_2(t') - \partial_t F_1(t')]_{z=L}, \tag{53}$$

where

$$F_2(t') = \partial_z K_0\{(\kappa_{0m}/R)\sqrt{[a_1 + i(z - t')][a_2 - i(z + t')]}\},$$

$$F_1(t') = K_0\{(\kappa_{0m}/R)\sqrt{[a_1 + i(z - t')][a_2 - i(z + t')]}\},$$

and

$$t' = t - \hat{z} - R^2/2\hat{z}.$$

Using the identity $K_0'(z) = -K_1(z)$, the result (53) can be reduced to the following form:

$$\Psi(0,0,\hat{z},t) = -\frac{iJ_1(\kappa_{0m})R}{8\pi\hat{z}} K_1\left(\frac{\kappa_{0m}}{R}\sqrt{[a_1 + i(L - t')][a_2 - i(L + t')]}\right) \frac{a_2 - i(L + t')}{\sqrt{[a_1 + i(L - t')][a_2 - i(L + t')]}}. \tag{54}$$

At the center of the pulse (i.e., $\hat{z} - t + L = 0$), the wave amplitude becomes

$$\Psi(0,0,\hat{z},t) = -\frac{iJ_1(\kappa_{0m})R}{8\pi\hat{z}} K_1\left\{\frac{\kappa_{0m}}{R}\sqrt{\left(a_1 + i\frac{R^2}{2\hat{z}}\right)\left[a_2 - i\left(2L - \frac{R^2}{2\hat{z}}\right)\right]}\right\} \frac{a_2 - i(2L - R^2/2\hat{z})}{\sqrt{[a_1 + i(R^2/2\hat{z})][a_2 - i(2L - R^2/2\hat{z})]}}. \tag{55}$$

As discussed in Sec. III, one can choose the condition $\kappa_{0m}\sqrt{(a_1+iR^2/2\hat{z})a_2}/R \ll 1$. If, furthermore, $a_2 \gg L$, an asymptotic expansion for small arguments yields $K_1(z) \simeq z^{-1}$ (cf., Ref. 21). Under these restrictions, (55) becomes

$$\Psi(0,0,\hat{z},t) \simeq \frac{-iJ_1(\kappa_{0m})R^2}{4\pi\kappa_{0m}} \frac{1}{(2a_1\hat{z}+iR^2)}, \qquad (56)$$

or, after rearranging the terms,

$$\Psi(0,0,\hat{z},t) \simeq \frac{J_1(\kappa_{0m})}{4\pi\kappa_{0m}} \frac{R^2 e^{-i\phi}}{\sqrt{(2a_1\hat{z})^2+R^4}}, \qquad (57)$$

with $\phi = \tan^{-1}(R^2/2a_1z) + \pi/2$. This result is interesting because, for $a_1 \ll 1$, the part of the solution given by (57) does not decay for $2a_1\hat{z} \ll R^2$, and is given approximately by

$$\Psi(0,0,\hat{z},t) \simeq [J_1(\kappa_{0m})/4\pi\kappa_{0m}]\, e^{-i\phi} \qquad (58)$$

for

$$\hat{z} < R^2/2a_1 . \qquad (59)$$

Combining Eqs. (59) and (51), the range of validity of the approximate solution (57) can be stated as follows:

$$1000(R/\kappa_{0m}) < \hat{z} < R^2/2a_1 . \qquad (60)$$

Note that the lower bound in Eq. (60) may not be strong enough to ensure the validity of the condition $\kappa_{0m}\sqrt{(a_1+iR^2/2\hat{z})a_2}/R \ll 1$. This can occur when $\hat{z}$ is fairly small, in that case we can set the limit

$$(\kappa_{0m}/R)\sqrt{(a_1+iR^2/2\hat{z})a_2} < 0.1 ,$$

and for $a_1 \ll 1$, the range given by Eq. (60) has to be modified to

$$\kappa_{0m}^2 a_2/0.02 < \hat{z} < R^2/2a_1 . \qquad (61)$$

The values of the different parameters will determine the correct range. Choose, for example, $\kappa_{0m}/R \simeq O(1)$, $a_2 = 5$ m, and $a_1 \ll 1$. This choice of parameters is typical for a waveguide a few meters long, with an opening of radius equal to 1 m and with mainly the first mode excited. In this case, $1000R/\kappa_{0m} = \kappa_{0m}^2 a_2/0.02 \simeq 1000$ and both ranges are identical. On the other hand, if higher modes are excited, the range (61) is valid, while the range (60) is correct if a waveguide of a larger radius is used.

The solution given in (57) has the unusual feature of hiding the $\hat{z}$ parameter within the $(2a_1\hat{z})^2 + R^4$ term since, for a small a_1, the effect of $\hat{z}$ does not appear until $2a_1\hat{z}$ takes over the R^4 term. This feature is shared by the EDEPT solutions of Ziolkowski.[7,10] As mentioned earlier, such solutions contain certain parameters that can be adjusted in order to slow down their decay. (For the semi-infinite waveguide, the parameters a_1 and a_2 are related directly to physical quantities, e.g., the amplitude of the initial pulse and its bandwidth.) Such behavior is not possible for the more conventional solutions. A good example along this direction is the following solution to the infinite waveguide problem[24]

$$\Psi(\mathbf{r},t) = J_0\left(\frac{\kappa_{0m}\rho}{R}\right) J_0\left(\frac{\kappa_{0m}}{R}\sqrt{t^2-z^2}\right) u(t-z) . \qquad (62)$$

It consists of a pulse traveling in the positive z direction with amplitude $J_0(\kappa_{0m}\rho/R)$ at $z=t$. An analysis analogous to that followed earlier in this section results in Eq. (53) with

$$F_2(t') = \partial_z\left[J_0\left(\frac{\kappa_{0m}}{R}\sqrt{t'^2-z^2}\right) u(t'-z)\right]$$

and

$$F_1(t') = J_0\left(\frac{\kappa_{0m}}{R}\sqrt{t'^2-z^2}\right) u(t'-z) .$$

Using the identity $J_0'(z) = -J_1(z)$ and carrying out the differentiations, one obtains

$$\Psi(0,0,z,t) = \frac{J_1(\kappa_{0m})R^2}{2\kappa_{0m}\hat{z}}\Bigg[J_1\left(\frac{\kappa_{0m}}{R}\sqrt{t'^2-L^2}\right) \times u(t'-L)\frac{\kappa_{0m}}{R}\frac{(t'+L)}{\sqrt{(t'-L)(t'+L)}} + 2J_0\left(\frac{\kappa_{0m}}{R}\sqrt{t'^2-L^2}\right)\delta(t'-L)\Bigg] . \qquad (63)$$

If the peak of the pulse is taken at $t' = L$, and one recalls that $J_1(z)$ behaves as $z/2$ for small arguments, the above expression can be approximated around its peak as follows:

$$\Psi(0,0,z,t) = \frac{J_1(\kappa_{0m})R^2}{2\kappa_{0m}\hat{z}}\left(\frac{\kappa_{0m}^2}{R^2}2L + 2\delta(0)\right) . \qquad (64)$$

It is seen that the wave amplitude dies off as $\hat{z}^{-1}$, even for higher modes. The unusual behavior observed earlier in connection with our original problem does not show up in this case. It should be pointed out that the $\delta(0)$ term appears in (64) because of the discontinuity in the pulse amplitude at $z = t$.

V. DISCUSSION

It was demonstrated in this paper that pulses analogous to Brittinghamlike solutions can be realized physically. Such pulses were obtained using a novel bidirectional decomposition, a brief discussion of which was provided. (A more detailed account of this new approach can be found in Ref. 20.) The aforementioned pulses can be fairly localized in a waveguide and the parameters arising within the context of their solution can be related to physically meaningful quantities. Traveling down the waveguide, which is highly dispersive, these pulses have unusual decay patterns. Basically, the waveguide can be divided into three regions, with the center of the pulses undergoing no decay in the first, a logarithmic decay in the second, and an exponential decay in the last portion. When launched in free space, these pulses were shown to have finite energy, to be causal and to exhibit an unusual decay pattern. (A similar behavior is not exhibited by other, more conventional solutions even if higher modes are used.)

Although the decay properties of the pulses studied here are similar to Wu's electromagnetic missiles,[25] the approach adopted in our work is quite different, mainly because it is of the Brittingham–Ziolkowski type. There are additional differences: (1) the missile solutions are restricted to TE_{01} modes only, whereas the solutions presented in this paper can incorporate a larger number of modes; (2) Wu analyzed the Poynting vector integrated over time and over the surface of a receiver located at variable distances, while in our analysis we dealt with peak field intensities along the line of sight of the aperture; (3) our pulses are initiated in a Cauchy initial-value fashion, with $\Psi(\mathbf{r},0)$ and $\Psi_t(\mathbf{r},0)$ specified, while in Wu's work the pulses are initiated with a rising edge proportional to $t^{\epsilon/2}$, with $\epsilon < 1$. The parameter ϵ is related directly to the slowness of the energy decay of the missiles. As a consequence, a pulse with a sharper rising edge will give a missile that decays at a slower rate as the distance to the receiver increases; in contrast, our solutions pertain to bounded distances depending primarily on the parameter a_1. In spite of these differences, it seems that the two approaches are pointing to the same underlying physical reality that needs to be investigated further.

The usual R^{-1} decay of a signal in the far field of an antenna calls upon defining some limit that separates the far from the near field. A very popular candidate is the Fresnel limit which is defined as $2D^2/\lambda$, where D is a characteristic dimension of the aperture and λ is a characteristic wave length of the signal. Such a definition has been developed primarily for monochromatic or modulated signals with a very narrow bandwidth. In the case of time-limited pulses, a characteristic wavelength λ has no meaning since, by definition, such pulses have virtually an infinite bandwidth. As a consequence, the notion of a Fresnel limit is very ambiguous, and a clear distinction between a far and a near field is not possible. Because of such ambiguities one can only compare different types of pulses with each other. Our work has shown that certain types of pulses can spread much more slowly than others as they propagate in free space. These results point out to the fact that techniques and ideas used to handle the transmission of cw signals might not be adequate in cases where time-limited pulses are involved, and call for the undertaking of a serious theoretical and experimental study of the propagation of pulses in free space. The hope is to establish that one can physically generate certain time-limited pulses which spread out at a much slower rate than cw signals. This work is a step towards such a goal.

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