$(ds')^2=(dx')^2+(dy')^2+(dz')^2+c^2(dt')^2$ in the frame $S'$. The transformation of coordinates that guarantees the invariance of the square of the interval, i.e., $(ds^2)=(ds')^2$, is given precisely by Eqs. (11). In this Euclidean space I can define geometrical objects as four vectors: $\mathcal{A}^\alpha$, $\mathcal{B}_\alpha$,..., and tensors: $\mathcal{W}^{\alpha\beta}$, $\mathcal{F}^\alpha_\nu$,... . An attractive feature of this space is that there is no difference between covariance and contravariance,[4] e.g., $\mathcal{A}^\alpha=\mathcal{A}_\alpha$, and $\partial^\alpha=\partial_\alpha$. With a bit of manipulation, I can write my Eqs. (7) as two tensor equations in this four space,[4]

$$\partial_\mu Q^{\mu\nu}=0, \quad \text{and} \quad \partial^\mu Q^{\nu\kappa}+\partial^\nu Q^{\kappa\mu}+\partial^\kappa Q^{\mu\nu}=0, \tag{17}$$

where $Q^{\mu\nu}$ is the electromagnetic field tensor in Euclidean four space, which is defined by the following antisymmetric matrix:

$$Q^{\mu\nu}=\begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}. \tag{18}$$

I can also write the tensor $Q^{\mu\nu}$ in terms of the four potential $\mathcal{A}^\mu=(-\Phi,\mathbf{A})$,

$$Q^{\mu\nu}=\partial^\nu\mathcal{A}^\mu-\partial^\mu\mathcal{A}^\nu, \tag{19}$$

where $\partial^\nu=[(1/c)(\partial/\partial t),\nabla]$. As you can see, Eq. (17) represent a respectable four-dimensional field theory. The expression (19) involves Eqs. (6) and so the functions $\mathbf{A}$ and $\Phi$ emerging from quantum or classical mechanics represent, by virtue of the elliptical propagation, the potentials of "my electromagnetism" and not that of Maxwell. In conclusion, your argument for deriving Maxwell's equations isn't as persuasive as you think; it's shown to be *ambiguous*. The really convincing things are always unambiguous, like *you* and *me*," finished the Devil with more than a hint of triumph."

And God offered a draw, which was immediately accepted.

[1]The present fiction has been inspired by a nice paper of Roger Barlow, "Introducing gauge invariance," Eur. J. Phys. **11**, 45–46 (1990). See also Andrzej Horzela, Edward Kapuscik, and Charles A. Uzes, "Comment on the paper 'Introducing gauge invariance,' by R. Barlow," Eur. J. Phys. **14**, 190 (1993).

[2]See, e.g., Donald H. Kobe, "Derivation of Maxwell's equations from the local gauge invariance of quantum mechanics," Am. J. Phys. **46**, 342–347 (1978).

[3]See, e.g., Donald H. Kobe, "Derivation of Maxwell's equations from the gauge invariance of classical mechanics," Am. J. Phys. **48**, 348–353 (1980); James S. Marsh, "Alternate 'derivation' of Maxwell's source equations from gauge invariance of classical mechanics," Am. J. Phys. **61**, 177–178 (1993); José A. Heras, "Comment on 'Alternate "derivation" of Maxwell's source equations from gauge invariance of classical mechanics,' by James S. Marsh [Am. J. Phys. **61**, 177–178 (1993)]," Am. J. Phys. **62**, 949–950 (1994).

[4]See, for example, E. Zampino, "A brief study on the transformation of Maxwell equations in Euclidean four-space," J. Math. Phys. **27**, 1315–1318 (1986).

# Electromagnetic field generated by a moving point charge: A fields-only approach

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There are various methods for obtaining the electromagnetic field generated by an arbitrarily moving point charge in free space. If one decomposes the total electric field in terms of its transverse and longitudinal components, one must deal with the fact that the longitudinal component is propagated instantaneously. In this paper we deal with this in a novel manner. In the Fourier transform domain we solve for the transform of the transverse electric field in terms of expressions which involve the transforms of the current density and the longitudinal electric field; we thus view these expressions as "sources" of the transverse electric field. By inverse Fourier transforming we directly obtain the space–time transverse electric field, which is shown to contain a term which exactly cancels the instantaneous longitudinal electric field, so that the total electric field is propagated in a retarded fashion. Our approach does not make use of intermediate vector and scalar potentials, and thus dispenses with the need for gauge conditions.

## I. INTRODUCTION

In this paper we solve for the electric field produced by a moving point charge in free space using a method that has not hitherto (to the best of our knowledge) been given. We first solve for the transverse electric field, in the Fourier transform domain, in terms of expressions involving the transforms of the current density and the longitudinal electric field; we thus view these last two quantities as "sources" of the transverse electric field, and we discuss the physical meaning of this interpretation. By applying an inverse Fourier transform we arrive, in a straightforward manner, at the space–time transverse electric field. A direct by-product of this analysis is a term which exactly cancels the instantaneously propagated longitudinal electric field. The total electric field is thus propagated in a retarded fashion. No intermediate vector and scalar potentials are used in this analysis.

The problem of finding the electric field produced by a moving point charge is traditionally solved by the introduction of intermediate vector and scalar potentials, but this immediately raises the issue of a gauge condition. The Liénard–Wiechert potentials are usually used to give the electric field for an electron moving with constant velocity;[1] an alternate approach in this special case is to apply a Lorentz transformation to the fields of a static charge. The electric field of an arbitrarily moving charge can also be found by the method of "normal variables" and Fourier transformation:[2] one proceeds by first finding the spatial Fourier transform of the transverse vector potential, which is gauge invariant. The causality of the electric field in the Coulomb gauge has been demonstrated by Brill and Goodman,[3] who show that the transverse current density, $\mathbf{J}_\perp(\mathbf{r},t)$, compensates for the instantaneous Coulomb interaction appearing when one calculates $\mathbf{E}$ in the Coulomb gauge. Also of interest is the work by Bohm and Weinstein[4] which considers nonradiating charge distributions using potentials and Fourier methods. We believe that the fields-only approach we present here may be preferred by some, due to its directness.

Using a method which is initially related to the one we use, Haus[5] derives the correct far field of an arbitrarily moving point charge in free space. He also attempts to explain the dichotomy between the current density, as being responsible for power transfer to the field according to Poynting's theorem, and the time derivative of the current density, as being responsible for far-field radiation. We discuss Haus' method, which seems flawed, in the final section of this paper.

## II. DERIVATION OF THE ELECTRIC FIELD

Our starting point is the vector equation obeyed by the electric field:

$$\nabla\times(\nabla\times\mathbf{E})+\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=-\mu_0\frac{\partial\mathbf{J}}{\partial t}, \tag{1}$$

where the vectors $\mathbf{E}\equiv\mathbf{E}(\mathbf{r},t)$ and $\mathbf{J}\equiv\mathbf{J}(\mathbf{r},t)$ are functions of the position $\mathbf{r}$ and the time $t$. We can apply a space-time Fourier transform to Eq. (1) to get

$$\mathbf{k}\times[\mathbf{k}\times\mathscr{E}]+\frac{\omega^2}{c^2}\mathscr{E}=-i\mu_0\omega\mathscr{J}, \tag{2}$$

where, for example,

$$\begin{aligned}\mathscr{E}\equiv\mathscr{E}(\mathbf{k},\omega)&\equiv\mathscr{F}_r\,\mathscr{F}_t\{\mathbf{E}\}(\mathbf{k},\omega)\\&=\int_{\mathrm{R}^3}d\mathbf{r}\int_{\mathrm{R}}dt\;e^{-i\mathbf{k}\cdot\mathbf{r}}e^{i\omega t}\mathbf{E}(\mathbf{r},t),\end{aligned} \tag{3}$$

and $k=|\mathbf{k}|=\sqrt{k_x^2+k_y^2+k_z^2}$. We may decompose $\mathscr{E}$ into components parallel and perpendicular to $\mathbf{k}$ as $\mathscr{E}=\mathscr{E}_\parallel+\mathscr{E}_\perp$, with the parallel components known as the (transform domain) *longitudinal field*, and the perpendicular component known as the *transverse field*. We have then that $\mathbf{k}\times\mathscr{E}=\mathbf{k}\times\mathscr{E}_\perp$, and so Eq. (2) decomposes into two equations: perpendicular to $\mathbf{k}$ and one parallel to $\mathbf{k}$.

Perpendicular to $\mathbf{k}$ we have

$$[k^2-(\omega^2/c^2)]\mathscr{E}_\perp=-i\omega\mu_0\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathscr{J})=i\mu_0\omega\mathscr{J}_\perp, \tag{4}$$

where the unit vector $\mathbf{k}/|\mathbf{k}|$ is denoted by $\hat{\mathbf{k}}$. Parallel to $\mathbf{k}$ we have

$$\frac{\omega^2}{c^2}\mathscr{E}_\parallel=-i\mu_0\omega\mathscr{J}_\parallel. \tag{5}$$

Since we write $\mathscr{J}_\perp=\mathscr{J}-\mathscr{J}_\parallel$, from Eq. (5) we have

$$\mathscr{J}_\perp=\mathscr{J}-\frac{i\omega}{\mu_0c^2}\mathscr{E}_\parallel, \tag{6}$$

and so Eq. (4) may be rewritten

$$[k^2-(\omega^2/c^2)]\mathscr{E}_\perp=i\mu_0\omega\mathscr{J}+\frac{\omega^2}{c^2}\mathscr{E}_\parallel. \tag{7}$$

It is tempting to divide Eq. (7) throughout by $k^2-\omega^2/c^2$, and then inverse Fourier transform the result to obtain $\mathbf{E}_\perp(\mathbf{r},t)$. However, some care must be exercised because, strictly speaking, this does *not* lead to a physically meaningful solution unless some caveats are followed. Even if the source current density, $\mathbf{J}(\mathbf{r},t)$, is spatially bounded, the longitudinal electric field, $\mathbf{E}_\parallel(\mathbf{r},t)$, whose space–time Fourier transform appears as a "source" term in Eq. (7) for $\mathscr{E}(\mathbf{k},\omega)$, is propagated instantaneously throughout space. This is readily seen from the Maxwell equation $\nabla\cdot\mathbf{E}_\parallel(\mathbf{r},t)=\rho(\mathbf{r},t)/\epsilon_0$, where $\mathbf{E}_\parallel(\mathbf{r},t)\Leftrightarrow\mathscr{E}_\parallel(\mathbf{k},\omega)$ form a space-time Fourier transform pair; any changes in $\rho(\mathbf{r},t)$ are manifested instantaneously throughout space in $\mathbf{E}_\parallel(\mathbf{r},t)$.

Nevertheless, if we impose the condition that the effects of both source terms on the right-hand side of Eq. (7) be propagated in a retarded sense in space–time, then the appropriate inverse spatial transform solution of Eq. (7) is given by

$$\begin{aligned}\mathscr{F}_t\{\mathbf{E}\}(\mathbf{r},\omega)=\frac{e^{i\omega r/c}}{4\pi r}*_{\mathbf{r}}\bigg(&i\omega\mu_0\mathscr{F}_t\{\mathbf{J}\}(\mathbf{r},\omega)\\&+\frac{\omega^2}{c^2}\mathscr{F}_t\{\mathbf{E}_\parallel\}(\mathbf{r},\omega)\bigg),\end{aligned} \tag{8}$$

where $*_{\mathbf{r}}$ denotes the operation of three-dimensional spatial convolution, and $r=|\mathbf{r}|$. This leads to the transverse space-time electric field,

$$\begin{aligned}\mathbf{E}_\perp(\mathbf{r},t)=&-\mu_0\frac{\partial}{\partial t}\int_{\mathrm{R}^3}d\boldsymbol{\xi}\,\frac{\mathbf{J}(\boldsymbol{\xi},t-|\mathbf{r}-\boldsymbol{\xi}|/c)}{4\pi|\mathbf{r}-\boldsymbol{\xi}|}\\&-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_{\mathrm{R}^3}d\boldsymbol{\xi}\,\frac{\mathbf{E}_\parallel(\boldsymbol{\xi},t-|\mathbf{r}-\boldsymbol{\xi}|/c)}{4\pi|\mathbf{r}-\boldsymbol{\xi}|}.\end{aligned} \tag{9}$$

This equation is central to our approach. It describes how $\mathbf{E}_\perp$ is manifested in terms of source components $\mathbf{J}$ and $\mathbf{E}_\parallel$.

Again, it should be emphasized that, although the longitudinal electric field, $\mathbf{E}_\parallel(\mathbf{r},t)$, is propagated instantaneously throughout space [via Eq. (5)] as a result of the parallel component of the current, $\mathbf{J}_\parallel(\mathbf{r},t)$, or of the charge density $\rho(\mathbf{r},t)$, the contributions of $\mathbf{E}_\parallel(\mathbf{r},t)$ from each point in space are in turn propagated causally (i.e., via the retarded propagator) to effect $\mathbf{E}_\perp(\mathbf{r},t)$ in Eq. (9). We shall demonstrate, in a novel fashion, that the second source term in Eq. (9) contains a component which exactly cancels the longitudinal electric field, $\mathbf{E}_\parallel$, at $\mathbf{r}$ at time $t$.

We now assume the existence of a lone particle of charge $q$ located in free space at the point $\mathbf{r}_0(t)$, so that $\rho(\mathbf{r},t)=q\delta[\mathbf{r}-\mathbf{r}_0(t)]$, and hence $\mathbf{J}(\mathbf{r},t)=q\mathbf{v}(t)\delta[\mathbf{r}-\mathbf{r}_0(t)]$, where $\mathbf{v}(t) \equiv d\mathbf{r}_0/dt$ is the velocity of the particle. All the subsequent analysis may readily be extended to cases of a finite or infinite number (including a continuum) of charged particles. Both source terms (integrals) in Eq. (9) contribute terms which are dependent on the velocity of the particle, and on the acceleration. We shall deal with the contributions from each integral separately.

In the second integral in Eq. (9) we make the change of variables $\boldsymbol{\zeta} = \mathbf{r} - \boldsymbol{\xi}$, and then convert the triple integral with respect to $\boldsymbol{\zeta}$ to spherical polar coordinates, to write

$$\int_{\mathrm{R}^3} d\boldsymbol{\xi}\, \frac{\mathbf{E}_\parallel(\boldsymbol{\xi}, t - |\mathbf{r}-\boldsymbol{\xi}|/c)}{4\pi|\mathbf{r}-\boldsymbol{\xi}|} = \frac{1}{4\pi}\int_0^\infty d\zeta\, \zeta \int_{\mathscr{S}^2} d\mathscr{S}^2 \times \mathbf{E}_\parallel(\mathbf{r} - \zeta\hat{\boldsymbol{\zeta}}, t - \zeta/c), \tag{10}$$

where $\zeta = |\boldsymbol{\zeta}|$, $\hat{\boldsymbol{\zeta}} = \boldsymbol{\zeta}/\zeta$, and $\int_{\mathscr{S}^2} d\mathscr{S}^2$ denotes the integration over the unit sphere, with respect to the unit vector $\hat{\boldsymbol{\zeta}}$. Making the further change of variable $\zeta/c = \tau$ allows us to rewrite Eq. (10)

$$\int_{\mathrm{R}^3} d\boldsymbol{\xi}\, \frac{\mathbf{E}_\parallel(\boldsymbol{\xi}, t - |\mathbf{r}-\boldsymbol{\xi}|/c)}{4\pi|\mathbf{r}-\boldsymbol{\xi}|} = \frac{c^2}{4\pi}\int_0^\infty d\tau\, \tau \int_{\mathscr{S}^2} d\mathscr{S}^2 \times \mathbf{E}_\parallel(\mathbf{r} - \hat{\boldsymbol{\zeta}} c\tau, t - \tau). \tag{11}$$

An expression for the longitudinal electric field can be obtained from Green's Representation Theorem[6] for a sphere $V$, of radius $c\tau$, centered on the point $\mathbf{r}$

$$E_{\parallel x}(\mathbf{r}, t-\tau) = \frac{-1}{4\pi}\int_V d\mathbf{r}'\, \frac{\nabla^2 E_{\parallel x}(\mathbf{r}, t-\tau)}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi(c\tau)}\int_S \nabla E_{\parallel x}(\mathbf{r}', t-\tau)\cdot d\mathbf{S}' + \frac{1}{4\pi(c\tau)^2}\int_S dS'\, E_{\parallel x}(\mathbf{r}', t-\tau), \tag{12}$$

where $E_{\parallel x}$ denotes the $x$ component of $\mathbf{E}_\parallel$, and $\int_S dS'$ denotes integration with respect to $\mathbf{r}'$ over the surface $S$. Using Gauss's Theorem, we may rewrite Eq. (12) as

$$E_{\parallel x}(\mathbf{r}, t-\tau) = \frac{-1}{4\pi}\int_V d\mathbf{r}'\, \frac{\nabla^2 E_{\parallel x}(\mathbf{r}', t-\tau)}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi(c\tau)}\int_V \nabla^2 E_{\parallel x}(\mathbf{r}, t-\tau)\, d\mathbf{r}' + \frac{1}{4\pi}\int_{\mathscr{S}^2} d\mathscr{S}^2\, E_{\parallel x}(\mathbf{r} - c\tau\hat{\boldsymbol{\zeta}}, t-\tau), \tag{13}$$

where we have converted the last integral in Eq. (12), with respect to $\mathbf{r}'$ over the spherical surface $S$ centered at $\mathbf{r}$, to a corresponding integral with respect to the unit vector $\hat{\zeta}$ over the surface of the unit sphere centered at the origin. We shall now combine Eq. (13), along with analogous equations for the $y$ and $z$ components, into a single vector equation. First we note that $\nabla\cdot\mathbf{E}_\parallel = \rho/\epsilon_0$ implies that $\nabla^2\mathbf{E}_\parallel = \nabla\rho/\epsilon_0$, where $\nabla^2\mathbf{E}_\parallel$ is taken to represent the vector consisting of the Laplacian operating on the Cartesian components of $\mathbf{E}_\parallel$ (this result may be verified by remembering that $\nabla\times\mathbf{E}_\parallel = 0$ so that $\nabla\times\nabla\times\mathbf{E}_\parallel = -\nabla^2\mathbf{E}_\parallel + \nabla(\nabla\cdot\mathbf{E}_\parallel) = 0$; hence, $\nabla(\nabla\cdot\mathbf{E}_\parallel) = \nabla^2\mathbf{E}_\parallel$). The vector analogue of Eq. (13) then becomes

$$\mathbf{E}_\parallel(\mathbf{r}, t-\tau) = \frac{-1}{4\pi\epsilon_0}\int_V d\mathbf{r}'\, \frac{\nabla\rho(\mathbf{r}', t-\tau)}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi\epsilon_0(c\tau)}\int_V d\mathbf{r}'\, \nabla\rho(\mathbf{r}', t-\tau) + \frac{1}{4\pi}\int_{\mathscr{S}^2} d\mathscr{S}^2\, \mathbf{E}_\parallel(\mathbf{r}', t-\tau). \tag{14}$$

Now, since $\rho(\mathbf{r}', t-\tau) = q\,\delta[\mathbf{r}' - \mathbf{r}_0(t-\tau)]$, we see that

$$\int_V d\mathbf{r}'\, \nabla\rho(\mathbf{r}', t-\tau) = q\int_V d\mathbf{r}'\, \nabla\delta[\mathbf{r}' - \mathbf{r}_0(t-\tau)] = 0. \tag{15}$$

However,

$$\frac{-1}{4\pi\epsilon_0}\int_V d\mathbf{r}'\, \frac{\nabla\rho(\mathbf{r}', t-\tau)}{|\mathbf{r}-\mathbf{r}'|} = \frac{-q}{4\pi\epsilon_0}\int_V d\mathbf{r}'\, \frac{\nabla\delta[\mathbf{r}' - \mathbf{r}_0(t-\tau)]}{|\mathbf{r}-\mathbf{r}'|} = \begin{cases} \dfrac{q}{4\pi\epsilon_0}\nabla_{\mathbf{r}'}\left.\dfrac{1}{|\mathbf{r}-\mathbf{r}'|}\right|_{\mathbf{r}'=\mathbf{r}_0(t-\tau)} & \text{if } \mathbf{r}_0(t-\tau)\in V \\ 0 \quad \text{otherwise} \end{cases}. \tag{16}$$

We realize that $\mathbf{r}_0(t-\tau)\in V$ only if $|\mathbf{r} - \mathbf{r}_0(t-\tau)| < c\tau$, which allows us to rewrite the spatial conditions on the result (10) in terms of $u[c\tau - |\mathbf{r} - \mathbf{r}_0(t-\tau)|]$, where $u$ denotes the unit step function. Rewriting Eq. (16) thusly, and combining the result along with Eq. (15), allows us to rewrite Eq. (14) as

$$\mathbf{E}_\parallel(\mathbf{r}, t-\tau) = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r} - \mathbf{r}_0(t-\tau)}{|\mathbf{r} - \mathbf{r}_0(t-\tau)|^3}\, u[c\tau - |\mathbf{r} - \mathbf{r}_0(t-\tau)|] + \frac{1}{4\pi}\int_{\mathscr{S}^2} d\mathscr{S}^2\, \mathbf{E}_\parallel(\mathbf{r} - c\tau\hat{\boldsymbol{\zeta}}, t-\tau). \tag{17}$$

From Eq. (17) we may substitute for the inner integrand on the right-hand side in Eq. (11) to rewrite the second source term on the right-hand side in Eq. (1) as

$$\frac{-1}{c^2}\frac{\partial^2}{\partial t^2}\int_{\mathrm{R}^3} d\boldsymbol{\xi}\, \frac{\mathbf{E}_\parallel(\boldsymbol{\xi}, t - |\mathbf{r}-\boldsymbol{\xi}|/c)}{4\pi|\mathbf{r}-\boldsymbol{\xi}|} = -\frac{\partial^2}{\partial t^2}\int_0^\infty d\tau\, \tau\mathbf{E}_\parallel(\mathbf{r}, t-\tau) + \frac{q}{4\pi\epsilon_0}\frac{\partial^2}{\partial t^2}\int_0^\infty d\tau\, \tau\, \frac{\mathbf{r} - \mathbf{r}_0(t-\tau)}{|\mathbf{r} - \mathbf{r}_0(t-\tau)|^3} \times u[c\tau - |\mathbf{r} - \mathbf{r}_0(t-\tau)|]. \tag{18}$$

We may further write

$$-\frac{\partial^2}{\partial t^2}\int_0^\infty d\tau\tau\mathbf{E}_\parallel(\mathbf{r}, t-\tau) = -\frac{\partial^2}{\partial t^2}[tu(t) *_t \mathbf{E}_\parallel(\mathbf{r}, t)] = -\delta(t) *_t \mathbf{E}_\parallel(\mathbf{r}, t) = -\mathbf{E}_\parallel(\mathbf{r}, t), \tag{19}$$

where $*_t$ denotes the operation of convolution with respect to $t$. This demonstrates that $\mathbf{E}_\perp(\mathbf{r}, t)$ contains a term which explicitly cancels the instantaneous longitudinal field.

Regarding the second integral on the right-hand side of Eq. (18), we shall make the change of variables $t-\tau=\nu$. The semi-infinite range of integration with respect to $\nu$ may be extended to infinity in both directions by including an appropriate step function in the integral. Thus

$$\frac{\partial^2}{\partial t^2}\int_0^\infty d\tau\,\tau\,\frac{\mathbf{r}-\mathbf{r}_0(t-\tau)}{|\mathbf{r}-\mathbf{r}_0(t-\tau)|^3}\,u[c\tau-|\mathbf{r}-\mathbf{r}_0(t-\tau)|]$$
$$=\frac{\partial^2}{\partial t^2}\int_R d\nu\,\frac{\mathbf{r}-\mathbf{r}_0(\nu)}{|\mathbf{r}-\mathbf{r}_0(\nu)|^3}\,(t-\nu)u[t-\nu]\times u[c(t-\nu)-|\mathbf{r}-\mathbf{r}_0(\nu)|]. \tag{20}$$

We may now take the time derivatives inside the integral on the right-hand side in Eq. (20). If we discard terms which are zero in the sense of generalized functions, such as $(t-\nu)\delta(t-\nu)$, for example, and also assume that $\mathbf{r}_0(t)\neq\mathbf{r}$, then we may rewrite Eq. (20) as

$$\frac{\partial^2}{\partial t^2}\int_0^\infty d\tau\,\tau\,\frac{\mathbf{r}-\mathbf{r}_0(t-\tau)}{|\mathbf{r}-\mathbf{r}_0(t-\tau)|^3}\,u[c\tau-|\mathbf{r}-\mathbf{r}_0(t-\tau)|]$$
$$=c\,\frac{\partial}{\partial t}\int_R d\nu\,\frac{\mathbf{r}-\mathbf{r}_0(\nu)}{|\mathbf{r}-\mathbf{r}_0(\nu)|^3}\,(t-\nu)u[t-\nu]\times\delta[c(t-\nu)-|\mathbf{r}-\mathbf{r}_0(\nu)|]+c\int_R d\nu\,\frac{\mathbf{r}-\mathbf{r}_0(\nu)}{|\mathbf{r}-\mathbf{r}_0(\nu)|^3}\times u[t-\nu]\delta[c(t-\nu)-|\mathbf{r}-\mathbf{r}_0(\nu)|]. \tag{21}$$

To evaluate the integrals on the right-hand side in Eq. (21), we shall make the change of variable $c(t-\nu)-|\mathbf{r}-\mathbf{r}_0(\nu)|=\lambda$, whereupon we have

$$\int_R d\nu\,\delta[c(t-\nu)-|\mathbf{r}-\mathbf{r}_0(\nu)|]=\frac{1}{c}\int_R\frac{\delta(\lambda)d\lambda}{1-(\mathbf{v}(\nu)\cdot[\mathbf{r}-\mathbf{r}_0(\nu)])/c|\mathbf{r}-\mathbf{r}_0(\nu)|}. \tag{22}$$

Evaluating the integrands in Eq. (21) when $\lambda=0$ will imply that $c(t-\nu)=|\mathbf{r}-\mathbf{r}_0(\nu)|$, which implies setting $\nu$ equal to the retarded time $\nu=t_R=t-|\mathbf{r}-\mathbf{r}_0(t_R)|/c<t$, whence $t-\nu\rightarrow|\mathbf{r}-\mathbf{r}_0(t_R)|/c$ and $u[t-t_R]=1$. The results may be compactly written if we introduce the vector

$$\mathbf{w}(t)=\mathbf{r}-\mathbf{r}_0(t), \tag{23}$$

running from the particle's position at time $t$ to the fixed field point. Thus Eq. (21) becomes

$$\frac{\partial^2}{\partial t^2}\int_0^\infty d\tau\,\tau\,\frac{\mathbf{r}-\mathbf{r}_0(t-\tau)}{|\mathbf{r}-\mathbf{r}_0(t-\tau)|^3}\,u[c\tau-|\mathbf{r}-\mathbf{r}_0(t-\tau)|]$$
$$=c\,\frac{\partial}{\partial t}\left\{\frac{1}{[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]}\frac{\hat{\mathbf{w}}(t_R)}{|\mathbf{w}(t_R)|}\right\}+\frac{1}{[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]}\frac{\hat{\mathbf{w}}(t_R)}{|\mathbf{w}(t_R)|^2}, \tag{24}$$

where $\hat{\mathbf{w}}=\mathbf{w}/|\mathbf{w}|$. To carry out the remaining differentiation in Eq. (24) we shall decompose $\partial/\partial t\equiv(\partial t_R/\partial t)\partial/\partial t_R$. If we differentiate the relation $t_R=t-|\mathbf{r}-\mathbf{r}_0(t_R)|/c$ with respect to $t$ and solve for $\partial t_R/\partial t$, we obtain

$$\frac{\partial t_R}{\partial t}=\frac{1}{[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]}. \tag{25}$$

The differentiation (now with respect to $t_R$) is readily carried out in Eq. (24); we make use of the following results:

$$\frac{d}{dt_R}\,\mathbf{w}(t_R)=-\mathbf{v}(t_R),$$
$$\frac{d}{dt_R}\,|\mathbf{w}(t_R)|=-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R), \tag{26}$$
$$\frac{d}{dt_R}\,\hat{\mathbf{w}}(t_R)=\frac{1}{|\mathbf{w}(t_R)|}\{-\mathbf{v}(t_R)+[\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)]\hat{\mathbf{w}}(t_R)\}.$$

Combining both resulting terms on the right-hand side of Eq. (24) allows us to rewrite it as

$$\frac{\partial^2}{\partial t^2}\int_0^\infty d\tau\,\tau\,\frac{\mathbf{r}-\mathbf{r}_0(t-\tau)}{|\mathbf{r}-\mathbf{r}_0(t-\tau)|^3}\,u[c\tau-|\mathbf{r}-\mathbf{r}_0(t-\tau)|]$$
$$=\frac{\hat{\mathbf{w}}(t_R)[c^2-v^2(t_R)]-c\mathbf{v}(t_R)[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]}{c^2|\mathbf{w}(t_R)|^2[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]^3}+\frac{[\dot{\mathbf{v}}(t_R)\cdot\hat{\mathbf{w}}(t_R)]\hat{\mathbf{w}}(t_R)}{c^2|\mathbf{w}(t_R)|[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]^3}, \tag{27}$$

where $v(t_R)=|\mathbf{v}(t_R)|$ and $\dot{\mathbf{v}}(t)=d\mathbf{v}/dt$.

From Eqs. (19) and (27) we may then rewrite the second term on the right-hand side in Eq. (9), via Eq. (18), as

$$\frac{-1}{c^2}\frac{\partial^2}{\partial t^2}\int_{R^3}d\boldsymbol{\xi}\,\frac{\mathbf{E}_\parallel(\boldsymbol{\xi},t-|\mathbf{r}-\boldsymbol{\xi}|/c)}{4\pi|\mathbf{r}-\boldsymbol{\xi}|}$$
$$=-\mathbf{E}_\parallel(\mathbf{r},t)+\frac{q}{4\pi\epsilon_0}\times\left\{\frac{\hat{\mathbf{w}}(t_R)[c^2-v^2(t_R)]-c\mathbf{v}(t_R)[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]}{c^2|\mathbf{w}(t_R)|^2[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]^3}+\frac{[\dot{\mathbf{v}}(t_R)\cdot\hat{\mathbf{w}}(t_R)]\hat{\mathbf{w}}(t_R)}{c^2|\mathbf{w}(t_R)|[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]^3}\right\}. \tag{28}$$

We shall now consider the first source term on the right-hand side in Eq. (9). Since the current density in this case is $\mathbf{J}(\mathbf{r},t)=q\mathbf{v}(t)\delta[\mathbf{r}-\mathbf{r}_0(t)]$, we may rewrite this term

$$-\mu_0\frac{\partial}{\partial t}\int_{R^3}d\boldsymbol{\xi}\,\frac{\mathbf{J}(\boldsymbol{\xi},t-|\mathbf{r}-\boldsymbol{\xi}|/c)}{4\pi|\mathbf{r}-\boldsymbol{\xi}|}$$
$$=-q\mu_0\frac{\partial}{\partial t}\int_{R^3}d\boldsymbol{\xi}\,\frac{\mathbf{v}(t-|\mathbf{r}-\boldsymbol{\xi}|/c)}{4\pi|\mathbf{r}-\boldsymbol{\xi}|}\times\delta[\boldsymbol{\xi}-\mathbf{r}_0(t-|\mathbf{r}-\boldsymbol{\xi}|/c)]. \tag{29}$$

To evaluate the right-hand side integral we shall make the change of variables

$$\zeta=\xi-\mathbf{r}_0(t-|\mathbf{r}-\xi|/c), \tag{30}$$

whereupon Eq. (28) may be rewritten

$$-\mu_0\frac{\partial}{\partial t}\int_{R^3}d\xi\frac{\mathbf{J}(\xi,t-|\mathbf{r}-\xi|/c)}{4\pi|\mathbf{r}-\xi|}$$
$$=-q\mu_0\frac{\partial}{\partial t}\int_{R^3}\left\{d\xi\left[1-\mathbf{v}\left(t-\frac{|\mathbf{r}-\xi|}{c}\right)\right.\right.$$
$$\left.\left.\cdot\frac{(\mathbf{r}-\xi)}{c|(\mathbf{r}-\xi)|}\right]^{-1}\right\}\frac{\mathbf{v}(t-|\mathbf{r}-\xi|/c)}{4\pi|\mathbf{r}-\xi|}\delta(\zeta). \tag{31}$$

The integrand on the right-hand side in Eq. (31) must be evaluated for $\zeta=0$, so that $t-|\mathbf{r}-\xi|/c\rightarrow t_R$, and $\xi\rightarrow r_0(t_R)$. We obtain, using the notation introduced in Eq. (23),

$$-\mu_0\frac{\partial}{\partial t}\int_{R^3}d\xi\frac{\mathbf{J}(\xi,t-|\mathbf{r}-\xi|/c)}{4\pi|\mathbf{r}-\xi|}$$
$$=-\frac{q\mu_0}{4\pi}\frac{\partial}{\partial t}\left\{\frac{\mathbf{v}(t_R)}{|\mathbf{w}(t_R)|[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]}\right\}. \tag{32}$$

The differentiation with respect to $t$ is best done by decomposing $\partial/\partial t\equiv(\partial t_R/\partial t)\partial/\partial t_R$, as before, and then making use of Eq. (25) and the relations (26). If we also write $\mu_0=1/(c^2\epsilon_0)$, we obtain eventually

$$-\mu_0\frac{\partial}{\partial t}\int_{R^3}d\xi\frac{\mathbf{J}(\xi,t-|\mathbf{r}-\xi|/c)}{4\pi|\mathbf{r}-\xi|}$$
$$=-\frac{q}{4\pi\epsilon_0}\left\{\frac{c[\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)]\mathbf{v}(t_R)-v^2(t_R)\mathbf{v}(t_R)}{c^3|\mathbf{w}(t_R)|^2[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]^3}\right.$$
$$\left.+\frac{c\dot{\mathbf{v}}(t_R)[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]+[\dot{\mathbf{v}}(t_R)\cdot\hat{\mathbf{w}}(t_R)]\mathbf{v}(t_R)}{c^3|\mathbf{w}(t_R)|[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]^3}\right\}. \tag{33}$$

We may now combine the expressions in Eqs. (28) and (33) to give $\mathbf{E}_\perp(\mathbf{r},t)$ in Eq. (9). Finally, since $\mathbf{E}=\mathbf{E}_\perp+\mathbf{E}_\parallel$ we obtain after simplification,

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\left\{\frac{[\mathbf{w}(t_R)-\mathbf{v}(t_R)|\mathbf{w}(t_R)|/c][1-v^2(t_R)/c^2]}{|\mathbf{w}(t_R)|^3[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]^3}\right.$$
$$\left.+\frac{[\dot{\mathbf{v}}(t_R)\cdot\hat{\mathbf{w}}(t_R)][\hat{\mathbf{w}}(t_R)-\mathbf{v}(t_R)/c]-\dot{\mathbf{v}}(t_R)[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]}{c^2|\mathbf{w}(t_R)|[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]^3}\right\}. \tag{34}$$

This last expression may be interpreted and simplified in the usual way as follows. From the definition of $\mathbf{w}(t)$ in Eq. (23), we see that $|\mathbf{w}(t_R)|/c$ is the time taken for a signal traveling with speed $c$ from the particle's position at the retarded time, $t_R$, to reach the field point $\mathbf{r}$ at time $t$. Now, *if* the particle were traveling with constant velocity $\mathbf{v}(t_R)$ for all time following $t_R$, then the position of the particle at time $t$ would be $\mathbf{r}_0(t_R)+\mathbf{v}(t_R)|\mathbf{w}(t_R)|/c$. From Fig. 1, we see that $\mathbf{w}(t_R)-\mathbf{v}(t_R)|\mathbf{w}(t_R)|/c$ is the vector going from this supposed position of the particle at time $t$ (i.e., assuming it traveled with constant velocity $\mathbf{v}(t_R)$ for all time following $t_R$) to the field point $\mathbf{r}$. We shall thus define the vector

$$\mathbf{W}(t)=\mathbf{w}(t_R)-\mathbf{v}(t_R)|\mathbf{w}(t_R)|/c, \tag{35}$$

as shown in Fig. 1. From Fig. 1, we then see that

$$\mathbf{w}(t_R)\times\mathbf{v}(t_R)=\mathbf{W}(t_R)\times\mathbf{v}(t_R)$$
$$|\mathbf{w}(t_R)|\sin\theta=|\mathbf{W}(t_R)|\sin\psi. \tag{36}$$

Using Eq. (36) it is not too difficult to show that the denominator of the first term on the right-hand side of Eq. (34) may be rewritten as

$$|\mathbf{w}(t_R)|^3[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]$$
$$=|\mathbf{W}(t_R)|^3\left[1-\frac{v^2(t_R)}{c^2}\sin^2\psi\right]^{3/2}. \tag{37}$$

It is also conventional to define a normalized velocity

$$\beta(t_R)\equiv\frac{\mathbf{v}(t_R)}{c}. \tag{38}$$

Making use of the triple product relation $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$, we may thus rewrite the numerator of the second term on the right-hand side of Eq. (34) as

$$c\hat{\mathbf{w}}(t_R)\times\{[\hat{\mathbf{w}}(t_R)-\boldsymbol{\beta}(t_R)]\times\boldsymbol{\beta}(t_R)\}. \tag{39}$$

Thus, from Eqs. (37) and (39), we may rewrite the electric field in Eq. (34) as

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\frac{\mathbf{W}(t_R)}{|\mathbf{W}(t_R)|^3}\frac{1-v^2(t_R)/c^2}{\{1-[v^2(t_R)/c^2]\sin^2\psi\}^{3/2}}$$
$$+\frac{q}{4\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{\hat{\mathbf{w}}(t_R)\times\{[\hat{\mathbf{w}}(t_R)-\boldsymbol{\beta}(t_R)]\times\dot{\boldsymbol{\beta}}(t_R)\}}{|\mathbf{w}(t_R)|[1-\boldsymbol{\beta}(t_R)\cdot\hat{\mathbf{w}}(t_R)]^3}, \tag{40}$$

recovering the well-known result.[7] The first, "short range" component of the electric field in Eq. (40) depends only on the velocity of the particle at the retarded time, $t_R$, and is directed towards the field point $\mathbf{r}$ from the position the particle would occupy had it been traveling with constant velocity $\mathbf{v}(t_R)$ following $t_R$. The "far field" second component of

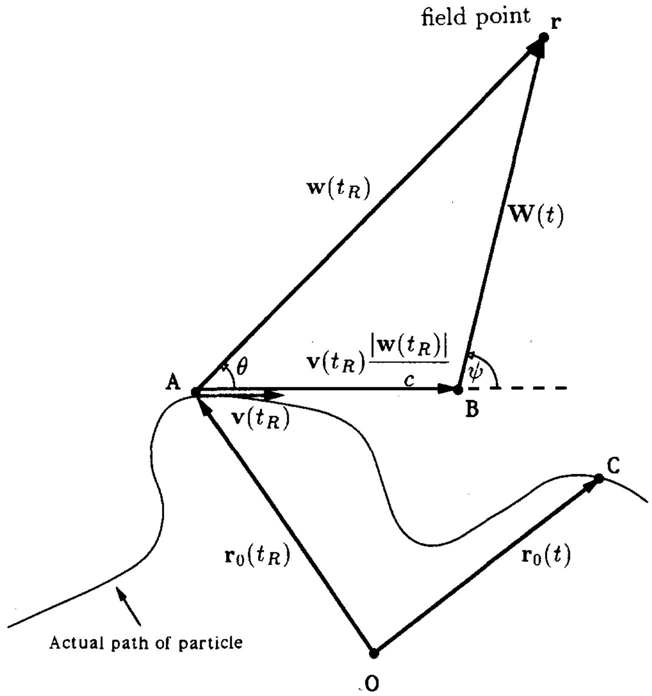


Fig. 1. The actual path of the particle is shown dotted. Point A is the position of the particle at the retarded time $t_R$ ; Point B is the position of the particle at time $t$ assuming that it continued travelling with velocity $\mathbf{v}(t_R)$ after time $t_R$ ; Point C is the actual position of the particle at time $t$.

the electric field is nonzero only if the particle has nonzero acceleration at $t_R$.

## III. DISCUSSION AND CONCLUSIONS

In the above work we have obtained the electric field due to a charged particle moving arbitrarily in free space; the method is applicable to arbitrary collections of particles. To obtain the electric field we first wrote the Fourier transform of the electric field [i.e., $\mathscr{E}(\mathbf{k},\omega)$] in terms of the transforms of the current density and the longitudinal electric field [in Eq. (7)]. We then applied an inverse Fourier transform to obtain an expression for the transverse electric field $\mathbf{E}_\perp(\mathbf{r},t)$ [in Eq. (9)]. This expression gives $\mathbf{E}_\perp(\mathbf{r},t)$ as the sum of two sources: the effect of the spatially bounded current density, propagating in a retarded fashion to the space–time field point $(\mathbf{r},t)$, and the effect of the longitudinal electric field, which exists throughout space instantaneously because of the charge density, but whose effects are nevertheless also propagated in a retarded fashion to the field point. Both source terms were explicitly evaluated, and were shown to contain effects due to both the velocity and the acceleration of the particle at the retarded time $t_R$ [Eqs. (28) and (33)]. It is worth commenting on the physical interpretation of the second source term in Eq. (9): the integral represents the weighted sum of contributions, to the point $\mathbf{r}$ at time $t$, coming from spherical layers centered on $\mathbf{r}$; as one proceeds outward from $\mathbf{r}$, one adds contributions from those $\mathbf{E}_\parallel$ values that existed in the layer at a preceding time, and the weighting on the contribution is inversely proportional to the distance of the layer from the center. We showed that the second time derivative of this weighted summation (integral) contains a quantity which exactly cancels the instantaneous longitudinal electric field, $\mathbf{E}_\parallel(\mathbf{r},t)$. This "Onion Theorem" result is represented by Eqs. (18) and (19). The total electric field is therefore propagated in a retarded manner. This novel approach did not make use of intermediate vector and scalar potentials.

Finally, we comment on a paper by Haus,[5] in which he derives the transverse electric far field of an arbitrarily moving point charge in free space, using methods which are initially similar to the ones we have used here. From an equation like (4) above, Haus solves for $\mathscr{E}_\perp(\mathbf{k},\omega)$ and then applies an inverse Fourier transform, converting the triple integral to spherical polar coordinates, to obtain essentially [see Eq. (10) in Ref. 4; we have used the notation developed above]

$$\mathscr{F}_t\{\mathbf{E}_\perp\}(\mathbf{r},\omega)=\frac{-i\omega\mu_0}{(2\pi)^3}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\int_0^{\infty}dk k^2 \times\frac{\hat{\mathbf{k}}\times[\hat{\mathbf{k}}\times\mathscr{J}(\mathbf{k},\omega)]}{k^2-(\omega/c)^2}\,e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{41}$$

Haus then writes "The last integral can be carried out by contour integration. For $\mathbf{k}\cdot\mathbf{r}>0$ the contour must be closed into the necessary imaginary half-plane of $\mathbf{k}$..." However, this reasoning seems flawed. In general $\mathscr{J}(\mathbf{k},\omega)$ will not satisfy the symmetry conditions that allow for extension of the $k$ integral to $-\infty$. Therefore, the contour integration results that Haus invokes are not valid. In any event, for the constant velocity moving point charge we have $\mathbf{J}(\mathbf{r},t)=q\mathbf{v}\delta[\mathbf{r}-\mathbf{r}_0(t)]$ so that[5] $\mathscr{J}(\mathbf{k},\omega)=2\pi q\mathbf{v}\delta(\omega-\mathbf{k}\cdot\mathbf{v})$. It is thus reasonable that the inner integral in Eq. (41) must be evaluated using generalized function means, i.e., it is the argument of the delta function which determines where the remainder of the integrand should be evaluated. It is incorrect to use contour integral methods to empower the real axis poles of the integrand to influence the behavior of the delta function term.

In the general case, for a particle moving with arbitrary velocity, it is possible to start with Eq. (41) and obtain the correct expression for the far field component of the transverse electric field using sounder methods than those used by Haus. We indicate such an approach in the Appendix.

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## APPENDIX

It is possible to obtain the correct expression for the far field component of the transverse electric field if one starts with Eq. (41), which we rewrite as

$$\mathscr{F}_t\{\mathbf{E}_\perp\}(\mathbf{r},\omega)=\frac{-i\mu_0\omega q}{(2\pi)^3}\int_{R^3}d\mathbf{k}\,\frac{\hat{\mathbf{k}}\times[\hat{\mathbf{k}}\times\mathscr{J}(\mathbf{k},\omega)]}{k^2-\omega^2/c^2}\,e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{A1}$$

without using the approach presented in the main text. As stated above, the current density in this case is $\mathbf{J}(\mathbf{r},t)=q\mathbf{v}(t)\delta[\mathbf{r}-\mathbf{r}_0(t)]$ from which we obtain [see Eqs. (3) and (4) in Ref. 4]

$$\mathscr{J}(\mathbf{k},\omega)=q\int_{\mathrm{R}}d\lambda\; e^{i\omega\lambda}\mathbf{v}(\lambda)e^{-i\mathbf{k}\cdot\mathbf{r}_0(\lambda)}. \tag{A2}$$

Substituting from Eq. (A2) into Eq. (A1) for $\mathscr{J}(\mathbf{k},\omega)$, and interchanging the order of integration, we obtain

$$\mathscr{F}_t\{\mathbf{E}_\perp\}(\mathbf{r},\omega)=\frac{-i\mu_0\omega q}{(2\pi)^3}\int_{\mathrm{R}}d\lambda\; e^{i\omega\lambda}\int_{\mathrm{R}^3}d\mathbf{k}\times\frac{\hat{\mathbf{k}}\times[\hat{\mathbf{k}}\times\mathbf{v}(\lambda)]}{k^2-\omega^2/c^2}\, e^{i\mathbf{k}\cdot[\mathbf{r}-\mathbf{r}_0(\lambda)]}. \tag{A3}$$

In Eq. (23) we defined $\mathbf{w}(\lambda)=\mathbf{r}-\mathbf{r}_0(\lambda)$. We can convert the inner triple integral in Eq. (A3) to spherical polar coordinates, with the north pole in the direction of $\mathbf{w}(\lambda)$, to rewrite it as

$$\int_0^\infty dk\;\frac{k^2}{k^2-\omega^2/c^2}\int_{\mathscr{S}^2}d\mathscr{S}^2\;\hat{\mathbf{k}}\times[\hat{\mathbf{k}}\times\mathbf{v}(\lambda)]e^{ik|\mathbf{w}(\lambda)|\cos\vartheta}. \tag{A4}$$

We are interested in a far field expression for $\mathbf{E}_\perp$, valid for large $|\mathbf{w}(\lambda)|$, and so we shall apply a stationary phase argument to Eq. (A4). The stationary (end) points are $\vartheta=0$ and $\vartheta=\pi$, at both of which $\hat{\mathbf{k}}\times[\hat{\mathbf{k}}\times\mathbf{v}(\lambda)]$ takes on the same value. The integral (A4) may thus be approximated by

$$\int_0^\infty dk\;\frac{k^2}{k^2-\omega^2/c^2}\;\hat{\mathbf{k}}\times[\hat{\mathbf{k}}\times\mathbf{v}(\lambda)]|_{\hat{\mathbf{k}}\|\mathbf{w}(\lambda)}\times\int_{\mathscr{S}^2}d\mathscr{S}^2e^{ik|\mathbf{w}(\lambda)|\cos\vartheta}\approx\frac{2\pi i}{|\mathbf{w}(\lambda)|}\;\hat{\mathbf{k}}\times[\hat{\mathbf{k}}\times\mathbf{v}(\lambda)]|_{\hat{\mathbf{k}}\|\mathbf{w}(\lambda)}\times\int_0^\infty dk\;\frac{k}{k^2-\omega^2/c^2}\;e^{ik|\mathbf{w}(\lambda)|\cos\vartheta}\;\Big|_0^\pi, \tag{A5}$$

once the integral over the unit sphere is performed on the rapidly oscillating exponential term. The restriction $|_{\hat{\mathbf{k}}\|\mathbf{w}(\lambda)}$ in Eq. (A5) means that the expression is to be evaluated for the unit vector $\hat{\mathbf{k}}$ parallel to the vector $\mathbf{w}(\lambda)$. Note that Haus writes [preceding Eq. (18) in Ref. 4] that for a corresponding integral over the unit sphere "the upper limit on $\theta$ is ignored because of the rapid variation of the exponent," which is incorrect. Evaluating the limits on $\vartheta$, we can then convert the semi-infinite integral with respect to $k$ to infinity in both directions, to rewrite Eq. (A5) as

$$\frac{-2\pi i}{|\mathbf{w}(\lambda)|}\;\hat{\mathbf{k}}\times[\hat{\mathbf{k}}\times\mathbf{v}(\lambda)]|_{\hat{\mathbf{k}}\|\mathbf{w}(\lambda)}\int_{-\infty}^{\infty}dk\;\frac{k}{k^2-\omega^2/c^2}\;e^{ik|\mathbf{w}(\lambda)|}. \tag{A6}$$

The real axis integration contour in Eq. (A6) may be deformed to capture only the pole at $k=\omega/c$ (this is not the only way to proceed),[8] corresponding to choosing a causal solution. With this, our expression (A3) for $\mathscr{F}_t\{\mathbf{E}_\perp\}(\mathbf{r},\omega)$ becomes

$$\mathscr{F}_t\{\mathbf{E}_\perp\}(\mathbf{r},\omega)\approx\frac{-i\mu_0\omega q2\pi^2}{(2\pi)^3}\int_{\mathrm{R}}d\lambda\;\frac{e^{i\omega\lambda}}{|\mathbf{w}(\lambda)|}\times\hat{\mathbf{k}}\times[\hat{\mathbf{k}}\times\mathbf{v}(\lambda)]|_{\hat{\mathbf{k}}\|\mathbf{w}(\lambda)}e^{i\omega|\mathbf{w}(\lambda)|/c}, \tag{A7}$$

valid for large $|\mathbf{w}|$.

If we apply an inverse temporal Fourier transform to Eq. (A7), and interchange the orders of integration, we get

$$\mathbf{E}_\perp(\mathbf{r},t)\approx\frac{-i\mu_0q2\pi^2}{(2\pi)^4}\int_{\mathrm{R}}d\lambda\;\frac{\hat{\mathbf{k}}\times[\hat{\mathbf{k}}\times\mathbf{v}(\lambda)]|_{\hat{\mathbf{k}}\|\mathbf{w}(\lambda)}}{|\mathbf{w}(\lambda)|}\times\int_{\mathrm{R}}d\omega\;\omega e^{-i\omega[t-\lambda-|\mathbf{w}(\lambda)|/c]}$$

$$=\frac{\mu_0q}{4\pi}\frac{\partial}{\partial t}\int_{\mathrm{R}}d\lambda\;\delta(t-\lambda-|\mathbf{w}(\lambda)|/c)\times\frac{\hat{\mathbf{k}}\times[\hat{\mathbf{k}}\times\mathbf{v}(\lambda)]|_{\hat{\mathbf{k}}\|\mathbf{w}(\lambda)}}{|\mathbf{w}(\lambda)|}. \tag{A8}$$

The integral in Eq. (A8) may be evaluated exactly as in Eqs. (21) and (22), and the eventual result is

$$\mathbf{E}_\perp(\mathbf{r},t)\approx\frac{\mu_0q}{4\pi}\frac{\partial}{\partial t}\frac{\hat{\mathbf{w}}(t_R)\times[\hat{\mathbf{w}}(t_R)\times\mathbf{v}(t_R)]}{[1-\mathbf{v}(t_R)\cdot\hat{\mathbf{w}}(t_R)/c]|\mathbf{w}(t_R)|}. \tag{A9}$$

The various derivatives in Eq. (A9) can be evaluated, using Eqs. (25) and (26), in a straightforward manner. If we retain only the leading terms (containing only an inverse power of $|\mathbf{w}(\lambda)|$, we further approximate Eq. (A9) as

$$\mathbf{E}_\perp(\mathbf{r},t)\approx\frac{\mu_0qc}{4\pi}\left[\frac{\hat{\mathbf{w}}(t)\times[\hat{\mathbf{w}}(t)\times\dot{\boldsymbol{\beta}}(t)]\{1-\boldsymbol{\beta}(t)\cdot\hat{\mathbf{w}}(t)\}}{\{1-\boldsymbol{\beta}(t)\cdot\hat{\mathbf{w}}(t)\}^3|\mathbf{w}(t)|}+\frac{\hat{\mathbf{w}}(t)\times[\hat{\mathbf{w}}(t)\times\boldsymbol{\beta}(t)](\dot{\boldsymbol{\beta}}(t)\cdot\hat{\mathbf{w}}(t))}{\{1-\boldsymbol{\beta}(t)\cdot\hat{\mathbf{w}}(t)\}^3|\mathbf{w}(t)|}\right]_{t=t_R}, \tag{A10}$$

where $\boldsymbol{\beta}$ is the normalized velocity defined in Eq. (38). Finally, the numerator in Eq. (A10) can be rearranged to show equivalence with that of the second term in Eq. (34), and so, following the working there, we have

$$\mathbf{E}_\perp(\mathbf{r},t)\approx\frac{q}{4\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{\hat{\mathbf{w}}(t_R)\times\{[\hat{\mathbf{w}}(t_R)-\boldsymbol{\beta}(t_R)]\times\dot{\boldsymbol{\beta}}(t_R)\}}{|\mathbf{w}(t_R)|[1-\boldsymbol{\beta}(t_R)\cdot\hat{\mathbf{w}}(t_R)]^3}, \tag{A11}$$

as the far field component of $\mathbf{E}_\perp(\mathbf{r},t)$ in Eq. (40).


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