# Nondispersive accelerating wave packets

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Motivated by the work of Berry and Balazs and Greenberger on the 1-D Schrödinger equation, we have investigated a class of nonspreading solutions to the 3-D Schrödinger equation involving accelerating Airy envelopes. These solutions are characterized by an asymmetric structure, in contrast to recently derived spherically symmetric packets moving with constant velocities. The field of a characteristic Airy packet extends in an oscillatory fashion behind its peak amplitude, while it quickly disappears in front of the packet's center. A particle modeled by such a packet seems to leave a wake of its field behind as it accelerates in a certain direction. On the other hand, a wave packet moving with a constant velocity has a field which is symmetrically distributed in all directions. Our work on Airy-type solutions to the 3-D Schrödinger equation has led us also to analogous solutions for the 3-D scalar wave equation.

## I. INTRODUCTION

It was realized by Berry and Balazs[1] that the force-free 1-D Schrödinger equation

$$i\hbar\,\partial_t\psi(x,t)+\frac{\hbar^2}{2m}\,\partial_x^2\psi(x,t)=0 \tag{1.1}$$

has a unique nonspreading packet solution expressed in terms of the Airy function,[2] viz.,

$$\psi(x,t)=\mathrm{Ai}\left(\frac{B}{\hbar^{2/3}}\left\{x-\frac{B^3t^2}{4m^2}\right\}\right)\,e^{i(B^3t/2m\hbar)[x-(B^3t^2/6m^2)]}, \tag{1.2}$$

where $B$ is an arbitrary constant. The square of the envelope $|\psi|^2$ of this wave packet travels in free space without any spreading. It is interesting, also, to note that it is moving with a velocity equal to $B^3t/2m^2$ which increases linearly with time. This means that the Airy packet (1.2) is moving with a uniform acceleration even though the Schrödinger equation (1.1) is force-free. Berry and Balazs provided an explanation of this unusual behavior by resorting to an integral representation of Eq. (1.2) which is composed of a superposition of plane waves, viz.,

$$\psi(x,t)=\frac{\hbar^{2/3}}{2\pi B}\int_{-\infty}^{+\infty}dk\;e^{i[kx-(\hbar k^2t/2m)+(\hbar^2k^3/3B^3)]}. \tag{1.3}$$

Assigning a particle to each plane wave and using an analogy to ray theory, they argued that the Airy packet corresponds to an ensemble of an infinite number of particles. The straight trajectories of these particles in a space–time diagram are enveloped by a parabolic caustic. The curvature of the caustic embodies the acceleration of the classically allowed region, which corresponds to the point where $x=B^3t^2/4m^2$. Berry and Balazs illustrated, furthermore, that if the Airy packet is allowed to evolve not in free space, but in a linear potential

$$V(x,t)=\frac{B^3}{2m}\,x, \tag{1.4}$$

the resulting force will be just enough to overcome the packet's natural tendency to accelerate and will bring its center to rest.

An alternative interpretation was provided by Greenberger[3] who argued that the Airy packet can be used to represent a free nonrelativistic particle falling in a constant gravitational field. Using the generalized Galilean transformation from a frame $X$ to a frame $X'$ defined by the coordinate relations

$$x'=x+\xi(t)\quad\text{and}\quad t'=t, \tag{1.5}$$

Greenberger was able to change the forced Schrödinger equation

$$i\hbar\,\partial_t\chi(x,t)+\frac{\hbar^2}{2m}\,\partial_x^2\chi(x,t)+\frac{B^3x}{2m}\,\chi(x,t)=0 \tag{1.6}$$

to the force-free equation

$$i\hbar\,\partial_t\psi(x',t)+\frac{\hbar^2}{2m}\,\partial_{x'}^2\psi(x',t)=0. \tag{1.7}$$

The new function $\psi(x',t)$ is related to the wave function $\chi(x,t)$ through the transformation

$$\chi(x,t)=\psi(x',t)e^{-i(m/\hbar)[\dot\xi(t)x'-\int dt\,\dot\xi^2(t)/2]}. \tag{1.8}$$

In the last expression $\dot\xi(t)$ denotes the time derivative of the function $\xi(t)$; the latter is governed by the equation of motion

$$m\,\frac{d^2\xi(t)}{dt^2}=-\frac{B^3}{2m}. \tag{1.9}$$

The wave function $\psi(x',t)$ representing a free particle is just the Airy packet given in Eq. (1.2). The wave function $\chi(x,t)$ has been transformed from the initial uniformly gravitating frame of reference $X$ to a free falling frame denoted by $X'$.

  

This is essentially the equivalence principle which states that all forces disappear in a free-falling system; it explains Berry's and Balazs' accelerating wave packet solution to a force-free equation. Using this reasoning, Greenberger concluded that the Airy packet does not spread out because it is a stationary state in a uniformly accelerating reference frame, such as the one associated with a uniform gravitational field.

## II. NONDISPERSIVE ACCELERATING WAVE PACKET SOLUTIONS TO THE 3-D SCHRÖDINGER EQUATION

The work of Berry and Balazs discussed in Sec. I can be extended to derive Airy-type, nonspreading solutions to the 3-D Schrödinger equation

$$i\hbar\partial_t\psi(\mathbf{x},t)+\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x},t)=0. \tag{2.1}$$

This can be carried out by considering the Fourier decomposition

$$\psi(\mathbf{x},t)=\frac{\hbar^2}{(2\pi)^3B^3}\int_{-\infty}^{+\infty}dk_1\int_{-\infty}^{+\infty}dk_2\int_{-\infty}^{+\infty}dk_3 \times F(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\hbar|\mathbf{k}|^2t/2m)}, \tag{2.2}$$

where $|\mathbf{k}|^2=k_1^2+k_2^2+k_3^2$. A direct extension of the spectrum appearing in Eq. (1.3), namely,

$$F(\mathbf{k})=(\alpha_1\alpha_2\alpha_3)^{1/3}e^{i\hbar^2\sum_j\alpha_jk_j^3/3B^3} \tag{2.3}$$

yields the following nonspreading, accelerating 3-D wave packet:

$$\psi(\mathbf{x},t)=\left(\prod_j \mathrm{Ai}\left(B(\alpha_j\hbar^2)^{-1/3}\left\{x_j-\frac{B^3t^2}{4\alpha_jm^2}\right\}\right)\right) \times\exp\left\{i\sum_j(B^3t/2m\alpha_j\hbar) \times[x_j-(B^3t^2/6m^2\alpha_j)]\right\}, \tag{2.4}$$

where $j=1,2,3$. Such a solution corresponds to the generalization of the Greenberger type solution to the 3-D potential

$$\sum_j B^3x_j/2m\alpha_j.$$

Another type of solution to the 3-D Schrödinger equation can be derived directly from Eq. (1.3); specifically,

$$\psi(\mathbf{x},t)=\frac{\hbar^{2/3}}{2\pi B}\int_{-\infty}^{+\infty}dk\; e^{i[k\sum_j\gamma_jx_j-(\hbar k^2t/2m)+(\hbar^2k^3/3B^3)]}, \tag{2.5}$$

where the $\gamma_j$'s are the direction cosines of the 3-D $\mathbf{k}$ vector. Carrying out the integration over $k$, we get

$$\psi(\mathbf{x},t)=\mathrm{Ai}\left(\frac{B}{\hbar^{2/3}}\left\{\sum_j\gamma_jx_j-\frac{B^3t^2}{4m^2}\right\}\right) \times\exp\left\{i\,(B^3t/2m\hbar)\left[\sum_j\gamma_jx_j-(B^3t^2/6m^2)\right]\right\}. \tag{2.6}$$

A simple rotation of coordinates reduces such a solution to the 1-D one given in Eq. (1.2).

The solutions given in Eqs. (2.4) and (2.6) are characterized by an oscillatory behavior for negative envelope arguments. They rise to their maxima at points very close to the zeros of the envelope arguments and fall off exponentially to zero for positive arguments. For example, the solution given in Eq. (2.4) has its greatest value at a point very close to $x_j=B^3t^2/4\alpha_jm^2$, while its field extends out behind it in a direction opposite to the direction of propagation.

It is interesting to compare the solutions given in Eqs. (2.4) and (2.6) with recently derived nonspreading packets moving with constant velocities. One such solution to the 3-D Schrödinger equation (2.1) assumes the following form:[4]

$$\psi(\mathbf{x},t)=j_0\{\sqrt{2\mu}[\rho^2+x_3-pt/m)^2]^{1/2}\} \times e^{ipx_3/\hbar}e^{-i(p^2/2m)t/\hbar}e^{-i(m^2c)t/\hbar}. \tag{2.7}$$

Here $j_0$ denotes the zeroth order spherical Bessel function, $\rho=(x_1^2+x_2^2)^{1/2}$, $\mu=mc/\hbar$, and $p=mv_g$; $v_g=\alpha_0\hbar/m$, $\alpha_0$ being an arbitrary constant.

The accelerating Airy packets given in Eqs. (2.4) and (2.6) are characterized by an asymmetric structure, in contrast to the spherically symmetric, constant velocity solution given in Eq. (2.7). The field of a typical Airy packet extends in an oscillatory fashion behind its peak amplitude, while it quickly disappears in front of the packet's center. It seems that a particle modeled by such a packet leaves a wake of its field behind as it accelerates in a certain direction. On the other hand, a wave packet moving with a constant velocity [e.g., the one given in Eq. (2.7)] has a field which is symmetrically distributed in all directions.

## III. NONDISPERSIVE AIRY-TYPE SOLUTIONS TO THE 3-D SCALAR WAVE EQUATION

Our results in connection to the 3-D Schrödinger equation have motivated us to search for analogous solutions to the 3-D wave equation

$$[c^{-2}\partial_t^2-\nabla^2]\psi(\mathbf{x},t)=0. \tag{3.1}$$

The Brittingham ansatz[5]

$$\psi(\mathbf{x},t)=G(\rho,\zeta)e^{i\beta\eta}, \tag{3.2}$$

where $\zeta=x_3-ct$, $\eta=x_3+ct$ and $\beta$ is an arbitrary parameter, reduces the 3-D scalar wave equation (3.1) to the 2-D Schrödinger equation

$$i4\beta\;\partial_\zeta G(\rho,\zeta)+\nabla_T^2G(\rho,\zeta)=0. \tag{3.3}$$

Here $\nabla_T^2$ is the transverse Laplacian. For the 2-D Schrödinger equation we can derive an Airy packet solution analogous to that given in Eq. (2.4); specifically,

$$G(\rho,\zeta)=\mathrm{Ai}[2\beta(x_1-3\beta\zeta^2/4)] \times\mathrm{Ai}[2\beta(x_2-3\beta\zeta^2/4)]e^{i2\beta^2\zeta(x_1+x_2-2\beta\zeta^2/3)}. \tag{3.4}$$

It should be noted that the $\alpha_j$ coefficients of Eq. (2.4) are chosen here to be equal to unity. Using this result, in conjunction with the Brittingham ansatz (3.2), we arrive at the following Airy packet solution to the 3-D scalar wave equation:

$$\psi(\mathbf{x},t)=\mathrm{Ai}[2\beta(x_1-3\beta\zeta^2/4)]\mathrm{Ai}[2\beta(x_2-3\beta\zeta^2/4)] \times e^{i2\beta^2\zeta(x_1+x_2-2\beta\zeta^2/3)}e^{i\beta\eta}. \quad (3.5)$$

This solution is *bidirectional*: It consists of a plane wave moving in the negative $x_3$ direction with speed $c$ which is modulated by an Airy-type wave packet; the latter moves with speed $c$ in the positive $x_3$ direction and remains invariant under translations along this axis.

## IV. CONCLUDING REMARKS

Berry and Balazs have established that the accelerating Airy wave packet given in Eq. (1.2) is a unique nonspreading solution to the 1-D Schrödinger equation. In this paper, it has been shown that a host of solutions of this type can be found for the 3-D Schrödinger equation. Two such have been exhibited explicitly and their salient features have been compared to those of the recently derived spherically symmetric nondispersive packets moving with constant velocities.

Benefiting from Brittingham ansatz, which maps the 3-D scalar wave equation into a 2-D Schrödinger equation via a reduction of dimensionality, Airy-type, nonspreading solutions have been found also for the 3-D scalar wave equation. In contradistinction to those for the 3-D Schrödinger equation, these solutions are bidirectional.

The use of nondispersive wave packets in modeling particles (e.g., electrons, photons, etc.) emphasizes the importance of their localization over the finiteness of their total energy content. However, the 3-D Schrödinger and scalar wave equations are linear; as a consequence, the nondispersive Airy packet solutions derived in this paper have infinite total energy content. Finite energy solutions can be obtained by further superpositions over the free parameters entering into the Airy packet solutions. This approach has been applied successfully to the 3-D scalar wave equation and has yielded a large class of solutions characterized by a high degree of localization.[6] On the other hand, it has been demonstrated[7] that if the localization of particles is given more priority, one can circumvent the infinite energy problem by utilizing the energy of the central portions of the nondispersive wave packets. With an appropriate choice of parameters, these packets can be rendered bumplike, with very large amplitudes around their centers compared to their tails.

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### RECOGNIZING TRUTH

One of the most important things in this 'guess–compute consequences–compare with experiment' business is to know when you are right. It is possible to know when you are right way ahead of checking all the consequences. You can recognize truth by its beauty and simplicity. It is always easy when you have made a guess, and done two or three little calculations to make sure that it is not obviously wrong, to know that it is right. When you get it right, it is obvious that it is right—at least if you have any experience—because usually what happens is that more comes out than goes in. Your guess is, in fact, that something is very simple. If you cannot see immediately that it is wrong, and it is simpler than it was before, then it is right. The inexperienced, and crackpots, and people like that, make guesses that are simple, but you can immediately see that they are wrong, so that does not count. Others, the inexperienced students, make guesses that are very complicated, and it sort of looks as if it is all right, but I know it is not true because the truth always turns out to be simpler than you thought. What we need is imagination, but imagination in a terrible strait-jacket. We have to find a new view of the world that has to agree with everything that is known, but disagree in its predictions somewhere, otherwise it is not interesting. And in that disagreement it must agree with nature.

Richard Feynman, *The Character of Physical Law* (M.I.T. Press, Cambridge, 1967), p. 171.