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Asymptotic evaluation of high-frequency fields near a caustic: An introduction to Maslov's method

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It is well known that the geometrical optics approximation, generally valid for high-frequency fields, fails in the vicinity of a caustic. A systematic procedure of V. P. Maslov that remedies this situation will be reviewed in this paper. Maslov's method makes use of a representation of the geometrical optics field in the phase space $M = X \times K$, where a point $m = (x, \kappa)$ is a pair of a position vector $x \in X$ and a wave vector $\kappa \in K$. A Lagrangian submanifold of M, Λ, that lies in the dispersion surface and is a union of the phase space trajectories selected by the initial conditions is constructed. It can be considered as a global representation of the phase. The phase space amplitudes (half densities) satisfy transport equations defined along those trajectories in Λ. Since trajectories in M never form a caustic, a globally defined amplitude can be established on Λ. The field on X is related to the resultant field on Λ by the "canonical operator," an operator introduced by Maslov. It generates an integral form of the solution near a caustic that can be evaluated analytically, numerically, or with uniform asymptotic techniques. Away from the caustic it recovers the geometrical optics field. Alternatively, the phase space field can be projected on a hybrid space Y where some of the space coordinates have been replaced by the corresponding wave vector components. For any caustic point in X, one such hybrid space Y where this projection does not encounter a caustic exists. A geometrical optics field results in Y that is related to the original in X by an asymptotic Fourier transform. The solution in X near a caustic can be represented as the Fourier transform to X of that hybrid space geometrical optics solution. These techniques are illustrated with two simple but revealing problems: continuation of the field through a fold caustic in a linear layer medium and through a caustic with a cusp point in a homogeneous medium.

1. INTRODUCTION

The purpose of this article is to bring to the attention of radio engineers and scientists concerned with high-frequency wave propagation some methods, attributable in part to V. P. Maslov [*Maslov*, 1972; *Maslov and Fedoryuk*, 1981], that have been applied mainly to physics as a bridge between classical and quantum mechanics. Their main application is to evaluate the field near a caustic where geometrical optics (GO), even augmented by the geometrical theory of diffraction (GTD) [*Keller*, 1962], fail. No attempt will be made to prove all statements. Rather, we will illustrate their application by simple problems. We wish to emphasize ideas rather than rigor. Furthermore, although the method is quite general, only the two-dimensional wave equation will be considered here. The examples chosen do not demonstrate adequately the power of the method but are more transparent as illustrations of the various steps involved.

There are several relevant articles by mathematicians such as *Duistermäat* [1973], *Guillemin and Sternberg* [1977], and *Hörmander* [1971] and by physicists such as *Berry and Mount* [1972], *Percival* [1977], and *Voros* [1976], but they require a rather sophisticated mathematics background and treat problems of more relevance to quantum mechanics. The basic work of Maslov is reported in [*Maslov*, 1972], a book written in Russian and translated into French, and a more recent book by *Maslov and Fedoryuk* [1981]. A very readable article is that of *Kravtsov* [1968]. The present paper, in which we have borrowed two examples from *Kravtsov* [1968], gives a more thorough discussion of the basic tools of

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Maslov: the Lagrangian submanifold Λ, which resides in the phase space $M = X \times K$ and gives a global description of the phase, and the canonical operator that relates the field descriptions on Λ to those in the original problem space X or in a hybrid space Y, whose coordinates are a combination of some space coordinates from X and some wave vector coordinates from K. A more thorough review of Maslov's method is given in *Ziolkowski* [1980].

This paper is organized as follows. Geometrical optics (GO) is reviewed briefly in section 2 and is applied to our two examples: plane wave propagation in a linear layer medium and propagation near a cusp caustic in a homogeneous medium. In section 3 the phase space approach to geometrical optics is discussed. Hamilton's equations and the associated flow, the Lagrangian submanifold, and amplitude half densities are introduced, and their connection with standard GO quantities is made. The canonical operator and the resultant representation of the field are defined in section 4. Two alternate descriptions of that representation also are given. Maslov's method is then applied to the aforementioned examples. We summarize in section 5 the major elements of that approach.

2. GEOMETRICAL OPTICS

2.1. *General aspects*

The GO method seeks (in the presence of a large parameter k, which following *Leray* [1972] will be denoted by $\nu \equiv ik$) approximate solutions to a partial differential equation

$$\mathscr{P}(x, D_x)U(x) = 0 \tag{1}$$

where $x \in \mathscr{R}^n$ and

$$D_x = \nu^{-1}\frac{\partial}{\partial x} \equiv \nu^{-1}\partial_x \tag{2}$$

of the form

$$u(x) = e^{\nu\Phi(x)}A(x) \tag{3}$$

where the phase Φ is a slowly varying, real-valued function and the amplitude A is a slowly varying, complex-valued function. In electromagnetics the large parameter $k = 2\pi/\lambda$, λ being the wavelength, and in quantum mechanics, $k = 2\pi/h$, h being Planck's constant. The amplitude may be taken as a function of ν represented by an $(m + 1)$ term asymptotic expansion:

$$A(x, \nu) = \sum_{j=0}^{m} \nu^{-j}A_j(x) \tag{4}$$

This expression is sometimes multiplied by ν^{μ}, $\mu \in \mathscr{R}$, which is important only when combining several fields of this type with different μ. The GO or ray optical field (3) separates the phase and the amplitude. The problem of finding the approximate solution resulting from given sources then decomposes into two parts: (1) ray tracing, which defines the continuation of the phase independently of the amplitude, and (2) determination of the amplitude, which can be carried out by following intensity variations along each ray without regard to the solution on other rays.

The determination of the phase and the associated descriptions—phase front, rays, fields of wave vectors, or group velocity vectors—is strictly geometrical optics in its original meaning. We shall designate it by $\overline{\text{GO}}$. In physics this corresponds to classical mechanics, where the rays become the point or system trajectories. The amplitude transport is actually a higher-order construct.

The general problem considered here is a Cauchy-type or "continuation" problem. The values of the field $U(x_0) = u(x_0)$ on a surface $X_0 \subset X$ are given, and the function (3) satisfying (1) is to be found in X. It is assumed that no caustics intersect X_0 and that a sense of crossing of X_0 is given. Furthermore, the given field $u(x_0)$ is assumed to have the GO field form

$$u(x_0) = f(x_0) = e^{\nu\phi(x_0)}a(x_0) \tag{5}$$

and $u(x)$ is constructed in the same form (3), except in some regions (the vicinity of caustics) where a different representation is needed. Maslov's method is precisely designed to furnish such a representation.

Note that the GO solution breaks down at a caustic in two ways: continuation of the phase by ray tracing beyond the caustic and determination of the amplitude by the transport equation on a caustic. The former difficulty arises because the phase function solution of the eikonal equation is (generally) multivalued; the caustic coincides with the join of the branches of the phase function. The branch of the phase function changes as the phase is continued through a caustic, and the characteristic $\pi/2$ phase shifts result. GO fails to give a prescription for the choice of the branch on which the continuation should proceed, hence, of the phase shift. Amplitude transport fails at a caustic because the tube of rays in which the intensity is being conserved has zero cross section there; thus GO (incorrectly) predicts an infinite amplitude at a caustic.

2.2. *Examples*

To illustrate these methods, we shall consider two particular examples of continuation problems. In both cases the two-dimensional Helmholtz equation

$$\tfrac{1}{2}\{\Delta + \varepsilon(x, z)k^2\}U(x, z) = 0 \qquad (6)$$

(the factor 1/2 is included to simplify many of the results derived below) with boundary values given for $z = 0$,

$$X_0 = \{(x, z) \mid z = 0\} \qquad (7)$$

is considered. The first problem is the continuation in a linear layer medium, i.e., in a medium of relative permittivity,

$$\varepsilon(x, z) = 1 - \alpha z \qquad (8)$$

of a given field

$$u(x_0, 0) = e^{\nu \xi_0 x_0} \qquad (9)$$

The second problem examines the continuation in the homogeneous medium

$$\varepsilon(x, z) = 1 \qquad (10)$$

of the (aperture) field

$$u(x_0, 0) = e^{\nu x_0^2/2b} a(x_0) \qquad -\alpha \le x_0 \le \alpha \qquad (11)$$

The GO solution of (6) is determined readily. The GO solution of the more general equation (1) is given in Appendix B. Rewrite the Helmholtz operator as

$$\mathscr{P}(x, D_x) = \tfrac{1}{2}[D_x^2 + D_z^2 - \varepsilon(x, z)]$$

and apply it to the GO field (3). One obtains

$$\mathscr{P}(x, D_x)u(x) = e^{\nu\Phi}\big[\tfrac{1}{2}(\Phi_x^2 + \Phi_z^2 - \varepsilon)A + \nu^{-1}(\Phi_x A_x + \Phi_z A_z + \tfrac{1}{2}A\Delta\Phi) + \nu^{-2}\tfrac{1}{2}\Delta A\big] \qquad (12)$$

where the subscripts denote partial derivatives, for instance, $\Phi_x \equiv \partial_x \Phi$. The GO solution satisfies

$$\mathscr{P}(x, D_x)u(x) \sim \mathcal{O}(\nu^{-2}) \qquad (13)$$

its terms are generated from the equations determined by setting the coefficients of different powers of ν equal to zero. Thus the phase satisfies the eikonal (Hamilton-Jacobi) equation (terms independent of ν)

$$p(x, \Phi_x) = \tfrac{1}{2}[\Phi_x^2 + \Phi_z^2 - \varepsilon(x, z)] = 0 \qquad (14)$$

and the amplitude satisfies the transport equation (terms proportional to ν^{-1})

$$\Phi_x A_x + \Phi_z A_z + \tfrac{1}{2}A\,\Delta\Phi = 0 \qquad (15)$$

Obviously, the term $\nu^{-2}\,\Delta A$ is neglected. While the tracing of the rays in $X = \mathscr{R}^2$ can be done by solving a second-order differential equation

$$\partial_\tau^2 x = \nabla\left(\frac{n^2}{2}\right) = \partial_x\left(\frac{\varepsilon}{2}\right) \qquad (16)$$

this equation can be replaced by a system of first-order equations, the *Hamilton equations*,

$$\begin{cases}\partial_\tau x = \partial_\kappa p(x, \Phi_x) = \kappa = \Phi_x \\ \partial_\tau \kappa = -\partial_x p(x, \Phi_x) = \tfrac{1}{2}\varepsilon_x\end{cases} \qquad (17)$$

where the wave vector $\kappa = (\xi, \zeta) = \Phi_x = (\Phi_x, \Phi_z)$ and τ is a parameter along the rays. Here, τ is the arc length. Note that in a more general medium the ray vector $v \equiv \partial_\kappa p$ and the wave vector κ will not coincide. The allowed wave vectors κ at each point x are determined by the dispersion relation

$$p(x, \kappa) = \tfrac{1}{2}[\kappa^2 - \varepsilon(x)] = 0 \qquad (18)$$

Since

$$\frac{d}{d\tau}\Phi = \Phi_x \cdot \frac{dx}{d\tau} = \kappa^2 = \varepsilon \qquad (19)$$

the phase continuation along a ray is given by

$$\Phi(x) = \phi(x_0) + \int_{x_0}^{x} \kappa \cdot dx = \phi(x_0) + \int_0^\tau \varepsilon\, d\tau \qquad (20)$$

Furthermore, since the operator

$$\frac{d}{d\tau} = \frac{dx}{d\tau} \cdot \partial_x = p_\kappa \cdot \partial_x = \Phi_x \cdot \partial_x \qquad (21)$$

the transport equation (15) reduces to an ordinary differential equation along a ray:

$$\frac{d}{d\tau}A + \tfrac{1}{2}A\,\Delta\Phi = 0 \qquad (22)$$

Let the coordinates $\sigma = (\sigma_1, \ldots, \sigma_{n-1})$ parameterize the rays. A point on any ray can then be labeled by σ and τ such that the points $x = (\sigma, \tau)$ and $x_0 = (\sigma, 0)$. Let $J(\sigma, \tau) = dx\,dz/d\sigma\,d\tau$ be the Jacobian of the transformation from those ray coordinates (σ, τ) to the space coordinates (x, z). It can be shown [*Leray*, 1972; *Maslov*, 1972; *Ziolkowski*, 1980] that

$$\frac{d}{d\tau}(\ln J) = \partial_x \cdot [p_\kappa(x, \Phi_x)] = \operatorname{div} v = \Delta\Phi \qquad (23)$$

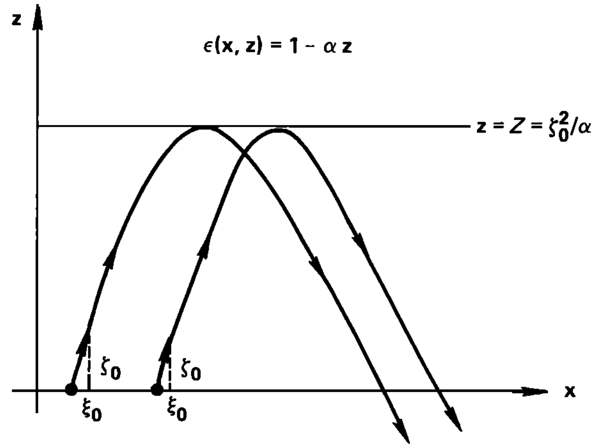

Fig. 1. The rays and the caustic in the linear layer problem.

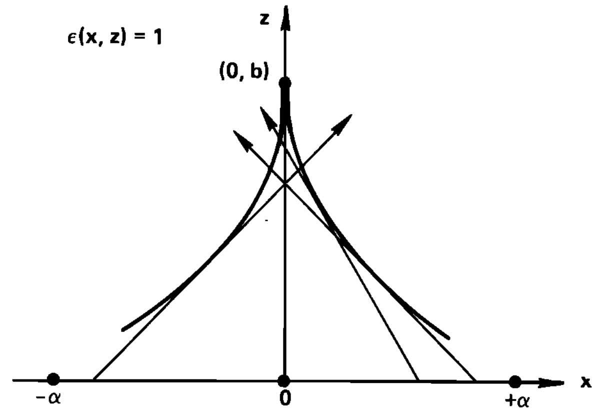

Fig. 2. The rays and the caustic in the homogeneous medium problem.

Therefore, if $A = J^{-1/2}\tilde{A}$, the amplitude $\tilde{A}$ satisfies

$$\frac{d}{d\tau}\tilde{A} = 0 \tag{24}$$

and the transport of the amplitude is defined as

$$A(\mathscr{x}) = \mathscr{D}^{-1/2}(\mathscr{x}/\mathscr{x}_0)a(\mathscr{x}_0) \tag{25}$$

where the divergence factor

$$\mathscr{D}^{-1/2}(\mathscr{x}/\mathscr{x}_0) = [J(\sigma, 0)/J(\sigma, \tau)]^{1/2} = [J(\mathscr{x}_0/\mathscr{x})]^{1/2} \tag{26}$$

Note that the divergence factor is defined by the behavior of the rays. Consequently, the GO field along a ray between $\mathscr{x}_0$ and $\mathscr{x}$ is

$$u(\mathscr{x}) = [\exp\left(\nu \int_{\mathscr{x}_0}^{\mathscr{x}} \kappa \cdot d\mathscr{x}\right)\mathscr{D}^{-1/2}(\mathscr{x}/\mathscr{x}_0)]u(\mathscr{x}_0) = G(\mathscr{x}, \mathscr{x}_0)u(\mathscr{x}_0) \tag{27}$$

If n rays whose initial points are $\mathscr{x}_{0j}$ pass through $\mathscr{x}$, the GO field is

$$u(\mathscr{x}) = \sum_{j=1}^{n} G(\mathscr{x}, \mathscr{x}_{0j})u(\mathscr{x}_{0j}) \tag{28}$$

The specific results for the linear layer (LLP) and the homogeneous medium (HMP) problems are listed in tabular form in Appendix A for quick reference. For both examples the rays can be labeled by x_0, i.e., $\sigma = x_0$. Note that $J(x_0, \tau) = 0$ at a caustic. In the LLP this occurs when $\tau = 2\zeta_0/\alpha$; thus, the straight line

$$z = \mathscr{Z} = \zeta_0^2/\alpha \tag{29}$$

is the caustic. The caustic in the HMP occurs when $\tau = b\zeta^2$. Combining this result, the ray equations, and the relation $x_0 = -b\xi$, the caustic points are ($x = b\xi^3$, $z = b\zeta^3$). Therefore, the caustic is described by the equation

$$\left(\frac{x}{b}\right)^{2/3} + \left(\frac{z}{b}\right)^{2/3} = 1 \tag{30}$$

It represents a cusp point at ($x = 0$, $z = b$). These caustics are illustrated in Figures 1 and 2.

3. PHASE SPACE APPROACH TO GEOMETRICAL OPTICS

A characteristic of Maslov's method is the emphasis on the representation of the GO field in the phase space $M = \mathscr{R}^{2n}$, where a pair $m = (\mathscr{x}, \kappa)$ consists of a position vector $\mathscr{x} \in X = \mathscr{R}^n$ and a wave vector $\kappa \in K = \mathscr{R}^n$ (momentum in physics). A connection be-

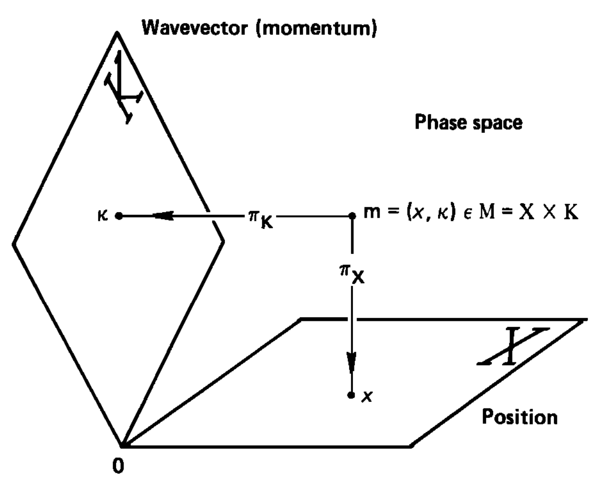

Fig. 3. Connections between the phase space and the position and wave vector spaces.

tween the phase space and the spaces X and K is provided by the projection maps

$$\pi_X : M \to X : m \mapsto x \tag{31}$$

$$\pi_K : M \to K : m \mapsto \kappa \tag{32}$$

They are depicted in Figure 3. In the examples, $x = (x, z)$, $\kappa = (\xi, \zeta)$, and $m = (x, z, \xi, \zeta)$.

3.1. *Dispersion relation*

Because of the slow variation of Φ and A, the GO field is locally a plane wave with propagation (wave) vector $\kappa = \nabla\Phi = \Phi_x$. This plane wave is a restriction to the particular values $\kappa = \Phi_x$ of the function $e^{\nu\kappa\cdot x}$ that is defined naturally on M with equal footing in both X and K space. For a particular problem the allowed values in M of κ for any x are determined by the dispersion relation

$$p(x, \kappa) = 0 \tag{33}$$

which is obtained by substituting the plane wave $e^{\nu\kappa\cdot x}$ into (1),

$$\mathscr{P}(x, D_x)e^{\nu\kappa\cdot x} = e^{\nu\kappa\cdot x}[p(x, \kappa) + \nu^{-1}p_1(x, \kappa) + O(\nu^{-2})] \tag{34}$$

and setting, according to $\overline{\text{GO}}$, the term independent of ν, $p(x, \kappa)$ (called the principal symbol of the operator $\mathscr{P}(x, D_x)$) to zero. The local projection of (33) onto X through the relation $\kappa = \Phi_x$ returns the eikonal equation (14).

The dispersion relation defines a $(2n - 1)$-dimensional hypersurface in M:

$$\mathfrak{S} = \{m \in M \mid p(x, \kappa) = 0\} \tag{35}$$

called the dispersion surface. For the LLP

$$\mathfrak{S} = \{(x, z, \xi, \zeta) \mid \xi^2 + \zeta^2 - (1 - \alpha z) = 0\} \tag{36}$$

and for the HMP

$$\mathfrak{S} = \{(x, z, \xi, \zeta) \mid \xi^2 + \zeta^2 - 1 = 0\} \tag{37}$$

3.2. *Hamilton's equations and flow*

A phase space approach to the $\overline{\text{GO}}$ problem is not new. In classical mechanics the trajectory of a point particle that results from integrating Newton's equation

$$m\frac{d^2x}{d\tau^2} = F \tag{38}$$

can be described by a first-order system in phase space, Hamilton's equations,

$$\begin{aligned} m\frac{dx}{d\tau} &= p = \partial_p H \\ \frac{dp}{d\tau} &= F = -\partial_x H \end{aligned} \tag{39}$$

where the Hamiltonian $H = (p^2/2m) + V$ and the force $F = -V_x$. Note that the LLP $\overline{\text{GO}}$ problem coincides with finding the trajectory of a point particle in a gravitational field (see Appendix C).

The Hamilton equations on M associated with the operator $\mathscr{P}(x, D_x)$ are defined by its principal symbol:

$$\begin{aligned} \frac{dx}{d\tau} &= p_\kappa(x, \kappa) \\ \frac{d\kappa}{d\tau} &= -p_x(x, \kappa) \end{aligned} \tag{40}$$

The principal symbol acts as the Hamiltonian of the $\overline{\text{GO}}$ problem. Hamilton's equations define a vector field

$$\mathscr{H}(m) = p_\kappa \cdot \partial_x - p_x \cdot \partial_\xi \tag{41}$$

that may be considered a velocity field

$$\partial_\tau m = (\partial_\tau x, \partial_\tau \kappa) = \mathscr{H}(m) = (p_\kappa, -p_x) \tag{42}$$

The integral curves of the Hamiltonian vector field $\mathscr{H}(m)$ (called bicharacteristic strips in the theory of partial differential equations) may be considered as phase space trajectories (rays). Their projections on the space X are the rays of geometrical optics; i.e., with the condition $\kappa = \Phi_x$, (17) is recovered. In particular, note that the projection of the velocity field (42) recovers the velocity field $v(x)$:

$$\pi_X : \partial_\tau m = (p_\kappa, -p_x) \mapsto \partial_\tau x = p_\kappa \tag{43}$$

The relationships are depicted in Figure 4.

Hamilton's equations are convenient for numerical integration. They have been used for ray tracing in the ionosphere [*Haselgrove*, 1955] and in magnetoactive plasmas [*Batchelor et al.*, 1980; *Bernstein*, 1975].

Another aspect of the Hamiltonian vector field $\mathscr{H}(m)$ provides further insight into the $\overline{\text{GO}}$ problem. A flow V^τ associated with a vector field V over a space N is a one-parameter group of diffeomorphisms (a map f is diffeomorphic if it is one-to-one and onto and if f and its inverse f^{-1} have continuous derivatives of all orders, i.e., are smooth.) of N into itself that is defined by the integral curves of V. The

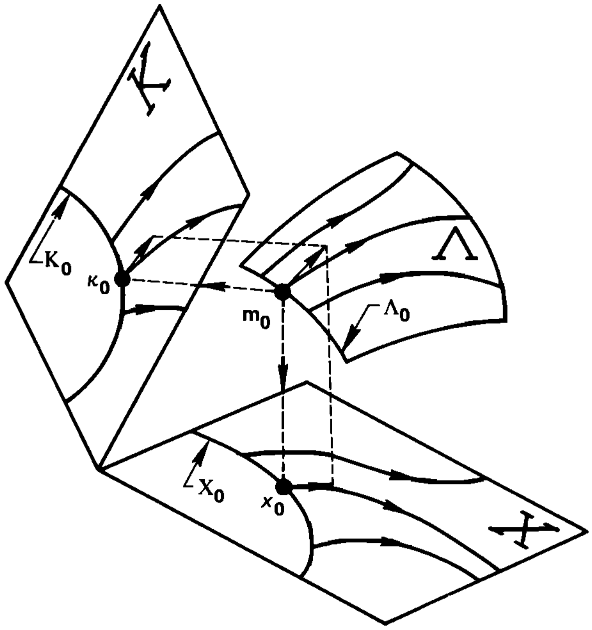

Fig. 4. The rays and their velocity vectors in the spaces X and K are the projections of the phase space trajectories and their velocity vectors.

Hamilton flow is the set of transformations, denoted by H^τ, of the phase space M into itself such that

$$\partial_\tau(H^\tau m) = \mathscr{H}(m) \tag{44}$$

and

$$H^0 m = m \tag{45}$$

i.e., a flow line $H^\tau m$ coincides with the trajectory through m. This flow has several important characteristics.

1. The principal symbol is invariant with respect to the transformations H^τ:

$$p(H^\tau m) = p(m) \tag{46}$$

2. The canonical two-form

$$\Omega = d\kappa \cdot dx = d\kappa^1 dx^1 + \cdots + d\kappa^n dx^n \tag{47}$$

is preserved by the flow

$$\Omega(H^\tau m) = \Omega(m) \tag{48}$$

This means the Hamilton flow is symplectic [*Arnol'd*, 1978].

3. The volume element $\omega = dx^N d\kappa^N = dx^1 \cdots dx^n d\kappa^1 \cdots d\kappa^n$ is preserved by the flow (Liouville's Theorem)

$$\omega(H^\tau m) = \omega(m) \tag{49}$$

Equivalently, the Hamilton flow is incompressible:

$$\operatorname{div} \mathscr{H} = \partial_x \cdot p_\kappa + \partial_\kappa \cdot (-p_x) \equiv 0 \tag{50}$$

Hence, if $\lambda_0 = m$ and $\lambda = H^\tau m$, then

$$\mathscr{D}^{-1/2}(\lambda/\lambda_0) = J^{1/2}(\lambda_0/\lambda) = 1 \tag{51}$$

Therefore, its trajectories do not form a caustic, i.e., the phase space rays are caustic free. This is a major reason for applying the phase space approach to GO. Note that the conservation properties (46), (48), and (49) are frequently expressed as

$$\mathfrak{L}_{\mathscr{H}} p = 0 \tag{46'}$$

$$\mathfrak{L}_{\mathscr{H}} \Omega = 0 \tag{48'}$$

and

$$\mathfrak{L}_{\mathscr{H}} \omega = 0 \tag{49'}$$

where $\mathfrak{L}_{\mathscr{H}}$ is the Lie derivative with respect to the vector field $\mathscr{H}$ [*Deschamps*, 1981, Appendix L].

3.3. *Lagrangian submanifold of M*

The construction of the phase function in the $\overline{\text{GO}}$ problem requires the introduction of an n dimensional Lagrangian submanifold of M. Note that a smooth n dimensional manifold Λ is a topological space that can be covered by a collection of open sets U_i, each of which can be mapped continuously by a smooth one-to-one (injective) function ϕ_i into a Euclidean space $\mathscr{R}^n$ and which satisfy the compatibility condition: on the overlap $U_i \cap U_j$ of two sets, the function $\phi_i \circ \phi_j$ defined on $\phi_j(U_i \cap U_j)$ is smooth. Roughly speaking, Λ locally "looks like" $\mathscr{R}^n$. The manifold Λ is a submanifold of a manifold N, provided there is a smooth injective mapping $\psi : \Lambda \to N$ whose derivative map $d\psi$ is also smooth and injective at each point of Λ. A submanifold Λ of M is isotropic if the canonical two-form Ω restricted to it is zero. If Λ is maximally isotropic (dim $M = 2$ dim Λ), then it is said to be a Lagrangian submanifold of M.

Now consider the phase function (or eikonal—the action in physics) Φ for a particular geometrical optics field. It defines the vector field $\kappa = \Phi_x$ as a function of x. The pairs (x, κ) lie on the graph Λ of that function:

$$\Lambda = \{(x, \kappa) \mid \kappa = \Phi_x(x)\} \tag{50}$$

This graph is an n dimensional Lagrangian submanifold of M(LSM) and Φ is its generating function.

Conversely, an n dimensional Lagrangian submanifold Λ of M defines locally a phase function $\Phi(x)$

under the following conditions:

1. Over a simply connected open region W_λ of Λ that contains λ, the projection π_X is diffeomorphic. Therefore, over the open region W_x of X, which is the projection (π_X) of W_λ, the diffeomorphism

$$\pi_X^{-1}: x \mapsto \lambda = (x, \kappa) \tag{51}$$

defines κ as a function of x:

$$\kappa = \sigma(x) \tag{52}$$

2. The one-form $\kappa \cdot dx$ is integrable over W_λ, i.e., there is a Φ such that $\kappa = \Phi_x$ only if on W_λ, $d(\kappa \cdot dx) = \Omega = 0$ or, equivalently, curl $\kappa|_\Lambda = 0$. This means a phase at λ_2 is related to a phase at λ_1 by

$$\Gamma(\lambda_2) = \Gamma(\lambda_1) + \int_{\lambda_1}^{\lambda_2} \kappa \cdot dx \tag{53}$$

where the path from λ_1 to λ_2 does not have to be specified. Letting $x_i = \pi_X(\lambda_i)(i = 1, 2)$, we define

$$\Phi(x_i) = \Gamma(\lambda_i) \tag{54}$$

Let those points of the manifold Λ at which the projection π_X is diffeomorphic be called regular points; the others being called singular points. Clearly, a neighborhood of a singular point Λ can no longer be generated by a phase function over X because the maps (51) and (52) are not defined there. A tangent plane to Λ at $\lambda = (x, \kappa)$ can be described as

$$T_\lambda \Lambda = \left\{ (x', \kappa') \in M \mid (\kappa' - \kappa)_i = \sum_{k=1}^{n} \sigma_{ik}(x' - x)_k \right.$$
$$\left. (x' - x)_j = \sum_{l=1}^{n} s_{jl}(\kappa' - \kappa)_l \right\} \tag{55}$$

where the indices i and j are elements of the sets I and J, respectively, that satisfy $I \cap J = \{0\}$ and $I \cup J = \{1, \ldots, n\}$ and where

$$\text{rank}\,(\sigma_{ik}) + \text{rank}\,(s_{jl}) = n \tag{56}$$

We call the term

$$N(\lambda) = n - \text{rank}\,(\sigma_{ik}) \tag{57}$$

the singularity index of λ. At a regular point λ_r, (52) gives rank $(\sigma_{ik}) = n$, hence $N(\lambda_r) = 0$. In contrast, $N(\lambda_s) \neq 0$ at a singular point λ_s. Therefore, at λ_s the tangent plane to Λ has become degenerate with respect to X and "vertical" (i.e., contains some directions that are parallel to K). We denote by Σ all of the singular points of Λ; Σ is called the singular set or apparent contour of Λ.

A submanifold Λ of M locally represents a solution of the eikonal equation $p(x, \Phi_x) = 0$ if and only if (1) $p(\lambda) = p(x, \kappa) = 0$ for all $\lambda \in \Lambda$, i.e., Λ lies in $\mathfrak{S}$; (2) Λ is a Lagrangian submanifold of M; (3) $N(\lambda) = 0$, where $x = \pi_X(\lambda)$. The obstructions to a global solution (i.e., a generating function Φ defined solely over X) are precisely those points at which (3) fails, i.e., the singular points. In fact, the caustic set of Λ

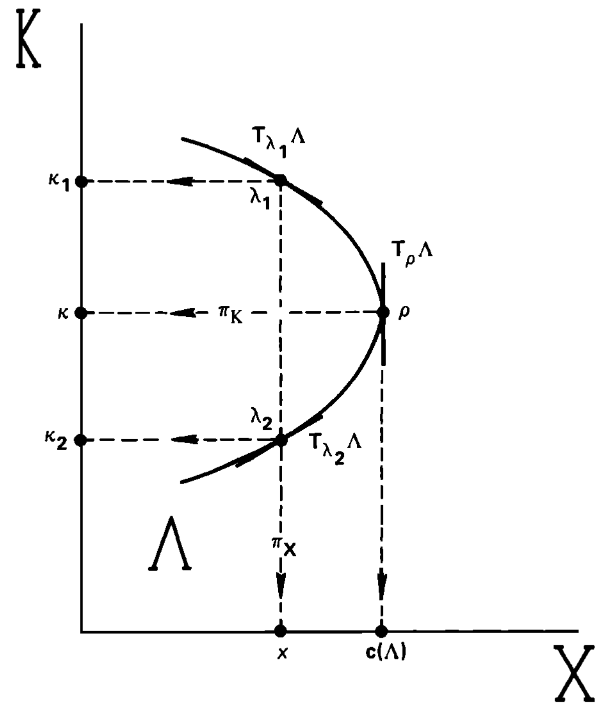

Fig. 5. Regular and singular points of a one-dimensional LSM Λ.

$$c(\Lambda) = \pi_X(\Sigma) \tag{58}$$

(the projection of Σ to X) coincides with the caustics of GO. Furthermore, the singular points can now be interpreted as the points at which the branches of Λ join together or, equivalently, where Λ bifurcates.

These features of a one-dimensional, two-branched LSM are shown in Figure 5. The points λ_1 and λ_2 of Λ are regular points. They both project to the same point x on X, but they belong to different branches of Λ. The tangent plane $T_\rho \Sigma$ at the singular point $\rho \equiv \Sigma$ is shown. It is vertical, i.e., parallel to K. The caustic point of Λ—$c(\Lambda)$—is indicated. The point ρ is clearly the join of the two branches of Λ. The degeneracy in the derivative of π_X at ρ is also apparent as x approaches $c(\Lambda)$.

Now reconsider Figure 5. Although Λ is not regular with respect to X near Σ, it is regular there with

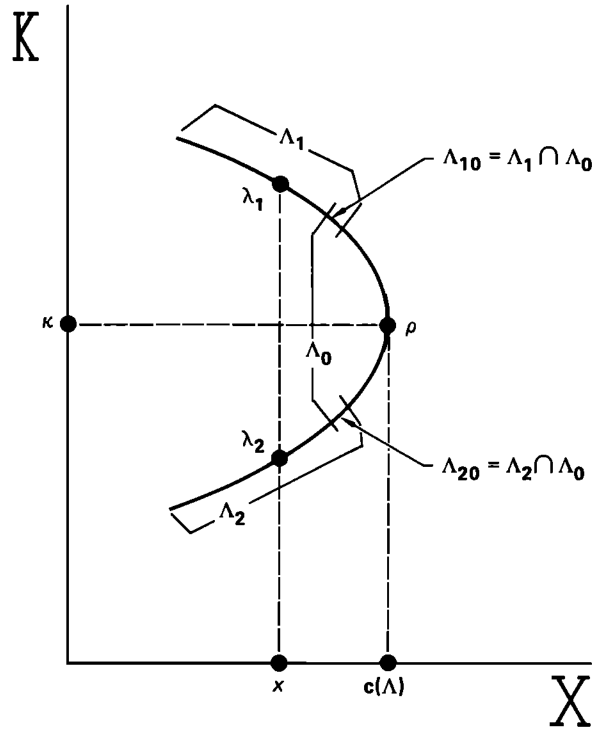

Fig. 6. Decomposition of a one-dimensional LSM Λ into regular and singular sets.

respect to K. Thus the diffeomorphism

$$\pi_K^{-1}: \kappa \mapsto (x, \kappa) \tag{59}$$

defines a neighborhood of ρ, W_ρ, as a function of κ,

$$x = s(\kappa) \tag{60}$$

and W_ρ can be generated by a phase function $\Psi(\kappa)$ defined over K as

$$W_\rho = \{(x, \kappa) \mid x = -\Psi_\kappa\} \tag{61}$$

If $\kappa_i = \pi_K(\lambda_i)(i = 1, 2)$, the relationship of the phases Γ and Ψ is

$$\Psi(\kappa_i) = \Gamma(\lambda_i) \tag{62}$$

where

$$\Gamma(\lambda_2) = \Gamma(\lambda_1) - \int_{\lambda_1}^{\lambda_2} x \cdot d\kappa \tag{63}$$

The minus sign is introduced so that $d(-x \cdot d\kappa) = \Omega$, the canonical two-form, in the same manner that $d(\kappa \cdot dx) = \Omega$. Consequently, by decomposing Λ into the "regular" sets Λ_1 and Λ_2 containing λ_1 and λ_2 and into the "singular" set Λ_0 containing ρ as shown in Figure 6, one can associate to Λ the phase functions $\{\Phi_1, \Phi_2, \Psi_0\}$ that generate those sets. The general case is treated in an analogous manner.

Let the point $x = (x, z)$, where $x = (x_1, \ldots, x_j)$ and $z = (z_1, \ldots, z_{n-j})$, and let the wave vector $\kappa = (\xi, \zeta)$, where $\xi = (\xi_1, \ldots, \xi_j)$ and $\zeta = (\zeta_1, \ldots, \zeta_{n-j})$. We call the space Y, whose coordinates $y = (\xi, z)$ are composed of $(n - j)$ components of the position vector x and j components of the wave vector κ corresponding to different directions, a hybrid space. A contribution by Maslov was the realization that, if an LSM Λ is not regular over X in a neighborhood W_ρ of a singular point ρ, there is at least one hybrid space Y with respect to which ρ is regular, i.e., with respect to which the projection map

$$\pi_Y: M \to Y: (x, \kappa) = (x, z, \xi, \zeta) \mapsto y = (\xi, z) \tag{64}$$

is diffeomorphic over W_ρ. Thus W_ρ can be defined through π_Y^{-1} as a function of y

$$\begin{cases} x = s(y) \\ \zeta = \sigma(y) \end{cases} \tag{65}$$

and generated by the phase function $\Psi(y)$ defined over Y as

$$W_\rho = \{(x, \kappa) \mid \zeta = \Psi_z(y) \quad x = -\Psi_\xi(y)\} \tag{66}$$

where if $y_i = \pi_Y(\lambda_i)(i = 1, 2)$,

$$\Psi(y_i) = \Gamma(\lambda_i) \tag{67}$$

and where

$$\Gamma(\lambda_2) = \Gamma(\lambda_1) + \int_{\lambda_1}^{\lambda_2} \zeta \cdot dz - x \cdot d\xi \tag{68}$$

Consequently, to a given LSM Λ, one can associate a set of phase functions $\{\Phi_r, \Psi_s\}$ that are defined over the spaces X and Y and that generate the regular and singular sets $\{\Lambda_r, \Lambda_s\}$ of Λ. This description is completed by defining how the phase functions Φ_r and Ψ_s fit together in an overlap region $\Lambda_r \cap \Lambda_s$.

Return once again to Figure 6. The overlap region $\Lambda_{10} = \Lambda_1 \cap \Lambda_0$ is described equivalently by the relations $\kappa = \sigma(x)$ and $x = s(\kappa)$ and is generated equivalently by the phase functions Φ_1 and Ψ_0 so that $\kappa = \Phi_{1x}$ and $-x = \Psi_{0\kappa}$. This means that over W_{10} the map $\sigma = s^{-1}$ and that the phase Ψ_0 is the Legendre transform [*Ziolkowski*, 1980; *Guckenheimer*, 1973, 1974*a*, *b*] of Φ_1 (denoted by $\Psi_0 = \mathscr{L}\Phi_1$) so that

$$\Psi_0(\kappa) = \Phi_1(s(\kappa)) - \kappa \cdot s(\kappa) \tag{69}$$

or equivalently, Φ_1 is the Legendre transform of Ψ_0 (denoted by $\Phi_1 = \bar{\mathscr{L}}\Psi_0$, where $\bar{\mathscr{L}} \equiv \mathscr{L}^{-1} = \mathscr{L}$) so that

$$\Phi_1(x) = \Psi_0(\sigma(x)) + x \cdot \sigma(x) \tag{70}$$

Similarly, in the general case the phase functions Φ_r and Ψ_s are related by a Legendre transformation in the overlap region $\Lambda_{rs} = \Lambda_r \cap \Lambda_s$.

Recall that the boundary values of the field, and hence the initial values of the GO field, are given in the form of a function $e^{\nu\phi}a$ on a surface $X_0 \subset X$. The initial wave vector

$$\kappa_0 = \Phi_x |_{X_0} \tag{71}$$

at $x_0 \in X_0$ is defined by the phase $\phi(x_0)$ of that function as follows. The function ϕ defines the gradient $\nabla_T \phi(x_0)$, which is the projection of κ_0 on the plane tangent to X_0 at point x_0. However, the vector κ_0 must belong to the dispersion surface $\mathfrak{S}$ at x_0, i.e., $p(x_0, \kappa_0) = 0$. Therefore, with the given sense of crossing of the field through X_0 (for instance, according to the side of X_0 occupied by the sources), the vector κ_0 is unambiguously determined. The pairs (x_0, κ_0) as x_0 describes X_0 generate a particular surface Λ_0 (an isotropic submanifold of dimension $n-1$) in the phase space M that lies in $\mathfrak{S}$. Since the principal symbol p and the canonical two-form Ω are invariants of the flow, the surface $\Lambda_\tau = H^\tau \Lambda_0$ also lies in $\mathfrak{S}$ and is isotropic. Therefore, the flow-out of Λ_0 through H^τ,

$$\Lambda = \bigcup_{\tau \geqslant 0} H^\tau \Lambda_0 \tag{72}$$

is n dimensional, hence an LSM, and lies in $\mathfrak{S}$. Consequently, the LSM Λ associated with a particular equation ($\mathscr{P}(x, D_x)$ defines $p(x, \kappa)$) and initial conditions (ϕ and sense of crossing with the dispersion relation define Λ_0) lies in $\mathfrak{S}$ and is the union of the phase space trajectories that originate in Λ_0. The latter description is depicted in Figure 4.

The LSM's in the examples are readily obtained from the equations for the phase space trajectories and the initial conditions. For both examples the initial submanifold

$$\Lambda_0 = \{m_0 = (x_0, 0, \xi_0, \zeta_0)\} \tag{73}$$

where

$$\xi_0 = \phi_{x_0}(x_0) = \begin{cases} \xi_0 & \text{LLP} \\ -x_0/b & \text{HMP} \end{cases} \tag{74}$$

and

$$\zeta_0 = \begin{cases} +(1-\xi_0^2)^{1/2} & |\xi_0| \le 1 \\ +i(\xi_0^2 - 1)^{1/2} & |\xi_0| > 1 \end{cases} \tag{75}$$

The LSM for the LLP is

$$\Lambda = \{m = (x, z, \xi, \zeta) \in M \mid \gamma(x, z, \xi) \equiv (\zeta^2(\xi)/\alpha) - (\mathscr{X} - z) = 0\} \tag{76}$$

Clearly, Λ lies in the dispersion surface $\mathfrak{S}$ defined by (36). For the HMP

$$\Lambda = \{m \in M \mid \gamma(x, z, \xi) \equiv x + b\xi - [\xi/\zeta(\xi)]z = 0\} \tag{77}$$

where the dispersion relation gives $\zeta(\xi) = (1 - \xi^2)^{1/2}$, the square root being defined as in (75). The tangent plane turns vertical (i.e., is parallel to a wave vector axis) in the LLP when $\gamma_\zeta = 0$, hence

$$\Sigma = \{m \in \Lambda \mid \zeta = [\alpha(\mathscr{X} - z)]^{1/2} = 0\} \tag{78}$$

and in the HMP when $\gamma_\xi = 0$, hence

$$\Sigma = \{m \in \Lambda \mid b - (z/\zeta^3) = 0\} \tag{79}$$

Upon elimination of the wave vector coordinates, the corresponding caustic sets are obtained:

$$c(\Lambda) = \{(x, z) \in X \mid z = \mathscr{X}\} \qquad \text{LLP} \tag{80}$$

$$c(\Lambda) = \{(x, z) \in X \mid (x/b)^{2/3} + (z/b)^{2/3} = 1\} \qquad \text{HMP} \tag{81}$$

3.4. *Index*

The $\overline{\text{GO}}$ description is completed by the introduction of an index that will determine the phase shift when a ray passes through a caustic. Although it is commonly associated with the amplitude, the index can be interpreted geometrically in terms of various properties of the LSM Λ. It was first interpreted in this manner by *Maslov* [1972], although Keller discussed its behavior in an analogous context earlier [*Keller*, 1958]. *Arnol'd* [1967] developed a description of the index in terms of the tangent planes of Λ rather than Λ itself. *Hörmander* [1971] later refined that description and modified its representation to conform with his Fourier integral operator expressions of the asymptotic solution. Our discussion will be slanted toward Maslov's original definition.

The singular set Σ has co-dimension [*Poston and Stewart*, 1978] + 1 in Λ (in the examples this generalizes the property that Σ is one-dimensional). Furthermore, a singular point ρ can be characterized by the condition that $\partial x_i/\partial\kappa_i = 0$ for some $i = \{1, \ldots, n\}$ ($T_\rho\Lambda$ is vertical over X and parallel to K along the direction κ_i). The term $\partial x_i/\partial\kappa_i$ has a different sign on both sides of Σ; hence the singular set Σ is orientable. This sign does not depend on the choice of variables x_i and κ_i. (Equivalently, these conditions mean that with an arbitrarily small defor-

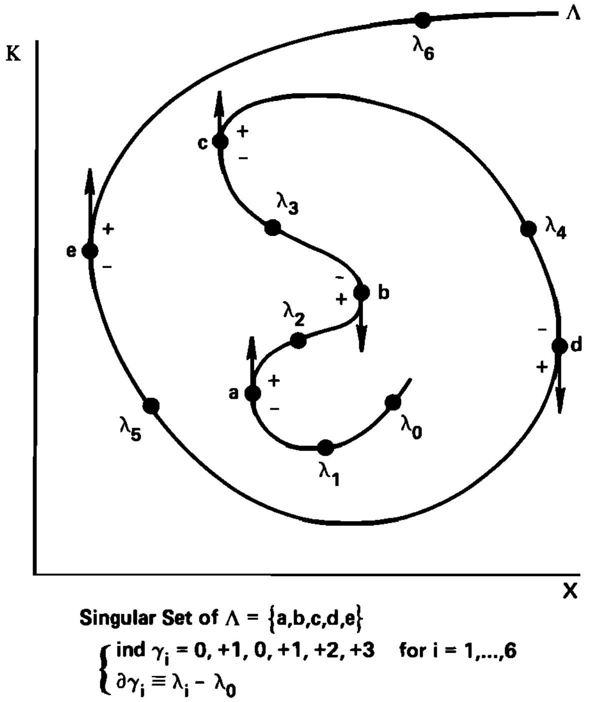

Fig. 7. Calculation of the index on a one-dimensional Lagrangian submanifold of M.

mation or rotation, Λ can be put into "general position" with respect to the projection map π_X.) The positive side of Σ is taken as the one on which $\partial x_i/\partial \kappa_i > 0$.

The index (*ind*) of an oriented curve γ whose endpoints are not contained in Σ and that does not coincide with Σ is defined to be the index of intersection of γ with Σ

$$ind\ \gamma = n_+ - n_- \tag{82}$$

where n_+ is the number of positive crossings of Σ (i.e., the number of times γ crosses Σ from the negative side to the positive side) and n_- is the number of negative crossings. Thus *ind* is a locally constant, integer-valued function on Λ.

A one-dimensional example that illustrates the calculation of the index is shown in Figure 7. The positive and negative sides of each point in $\Sigma = \{a, b, c, d, e\}$ are indicated, and a vector is shown that indicates the sense of positive crossing at these points. The indexes of the curves γ_i for $i = 1, \ldots, 6$ whose initial point is λ_0 and whose terminal points are λ_i are 0, +1, 0, +1, +2, +3.

The index *ind* can be used to define two separate indexes, ind and Ind, whose values can be identified with characteristics of the GO solution. Let Σ_τ be the component of Σ in Λ_τ. In the examples, Σ_τ is a set of distinct points. Let $W_{\tau i}$ and $W_{\tau j}$ be disjoint regular sets of Λ_τ, and let $\lambda_{\tau i}$ and $\lambda_{\tau j}$ be points in those sets. The index ind of a path $l[\lambda_{\tau i}, \lambda_{\tau j}]$ in Λ_τ, from $\lambda_{\tau i}$ to $\lambda_{\tau j}$ is defined as

$$\text{ind } l[\lambda_{\tau i}, \lambda_{\tau j}] = ind\ l[\lambda_{\tau i}, \lambda_{\tau j}] = n_+ - n_- \tag{83}$$

where now $n_+[n_-]$ is the number of positive [negative] crossings of Σ_τ by $l[\lambda_{\tau i}, \lambda_{\tau j}]$. If $\lambda_{\tau i}, \lambda_{\tau i'} \in W_{\tau i}$, then

$$\text{ind } l[\lambda_{\tau i}, \lambda_{\tau i'}] = 0 \tag{84}$$

and

$$\text{ind } l(\lambda_{\tau i}. \lambda_{\tau j}] = \text{ind } l[\lambda_{\tau i'}, \lambda_{\tau j}] \tag{85}$$

This index defines the relative values of the phases of the terms constituting the GO solution at a point of intersection of two or more rays. Similarly, the index Ind of the phase space trajectory $H^\tau\lambda_0$ through the points λ_0 and $\lambda_\tau = H^\tau\lambda_0$ is defined as

$$\text{Ind } H^\tau\lambda_0 = ind\ H^\tau\lambda_0 = ind\ l[\lambda_0, \lambda_\tau] \tag{86}$$

This index describes the phase shift in the GO solution as a ray passes through a caustic. It coincides with the Morse index [*Maslov*, 1972; *Arnol'd*, 1967].

As shown in [*Maslov*, 1972] and [*Arnol'd*, 1967], for any two trajectories $H^\tau\lambda_0$ and $H^\tau\lambda_0'$ the indexes ind and Ind satisfy the relation

$$\text{ind } l[\lambda_0', \lambda_0] + \text{Ind } H^\tau\lambda_0 = \text{Ind } H^\tau\lambda_0' + \text{ind } l[\lambda_\tau', \lambda_\tau] \tag{87}$$

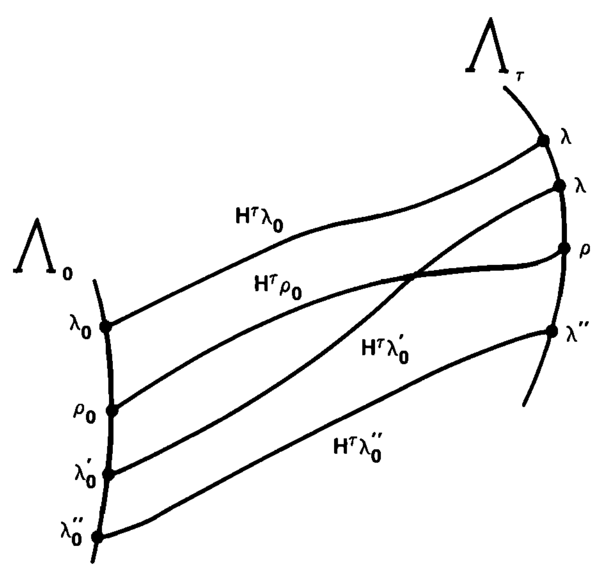

Fig. 8. Phase space trajectories connect points on the initial submanifold Λ_0 and its flow out $\Lambda_\tau = H^\tau\Lambda_0$.

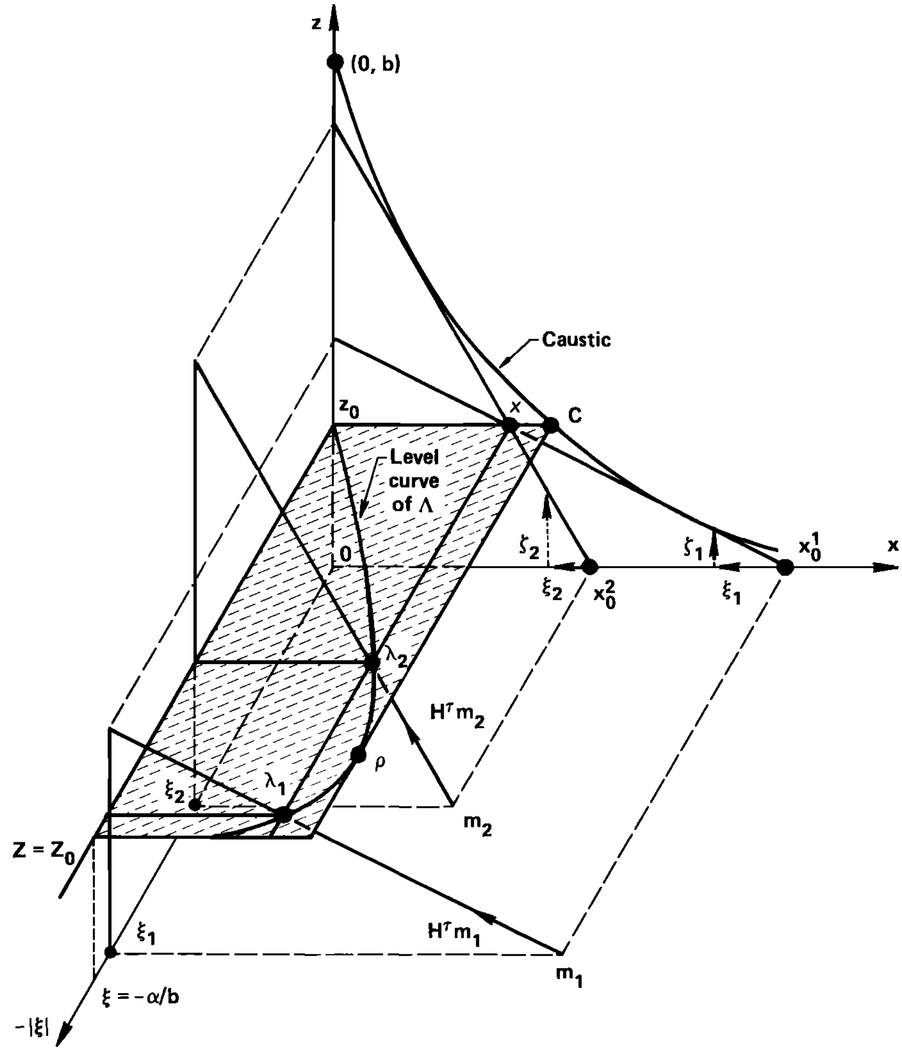

Fig. 9. The $z = z_0$ level curve of the LSM Λ.

where $\lambda_\tau = H^\tau\lambda_0$ and $\lambda'_\tau = H^\tau\lambda'_0$ in Λ_τ. This means the index of λ_τ relative to λ'_0 is independent of the path connecting them. Note that this correlates with the path independence of the definition of the phase function on Λ. Choosing a reference point λ^0 for the index in Λ_0, the index at a point λ in the examples reduces simply to the Morse index of the phase space trajectory $H^\tau\lambda_0$ through λ (i.e., since no caustics intersect X_0, Λ_0 itself is regular and ind $l[\lambda^0, \lambda_0] = 0$ for all $\lambda_0 \in \Lambda_0$). We denote this simply as

$$ind\ \lambda = \text{Ind}\ H^\tau\lambda_0 \tag{88}$$

The calculations of the indexes ind and Ind are illustrated in Figure 8. The trajectories through the pairs (λ_0, λ), (λ'_0, λ'), and (λ''_0, λ'') are $H^\tau\lambda_0$, $H^\tau\lambda'_0$, and $H^\tau\lambda''_0$, respectively. The trajectory $H^\tau\rho_0$ passes through the singular points ρ_0 and ρ and is a subset of Σ. One has ind $l[\lambda''_0, \lambda_0]$ = ind $l[\lambda'_0, \lambda_0] = 1$, ind $l[\lambda''_0, \lambda'_0] = 0$, Ind $H^\tau\lambda''_0$ = Ind $H^\tau\lambda_0 = 0$, ind $l[\lambda'', \lambda']$ = ind $l[\lambda'', \lambda] = 1$, ind $l[\lambda', \lambda''] = 0$, and Ind $H^\tau\lambda'_0 = 1$ such that

$$\text{ind}\ l[\lambda''_0, \lambda'_0] + \text{Ind}\ H^\tau\lambda'_0 = \text{Ind}\ H^\tau\lambda''_0 + \text{ind}\ l[\lambda'', \lambda']$$

and

$$\text{ind}\ l[\lambda''_0, \lambda_0] + \text{Ind}\ H^\tau\lambda_0 = \text{Ind}\ H^\tau\lambda''_0 + \text{ind}\ l[\lambda'', \lambda]$$

Since it indicates that a phase space trajectory has crossed the singular set, the index also marks the passage of a GO ray through a caustic. This is depicted in Figure 9 and 10. In Figure 9 a segment of

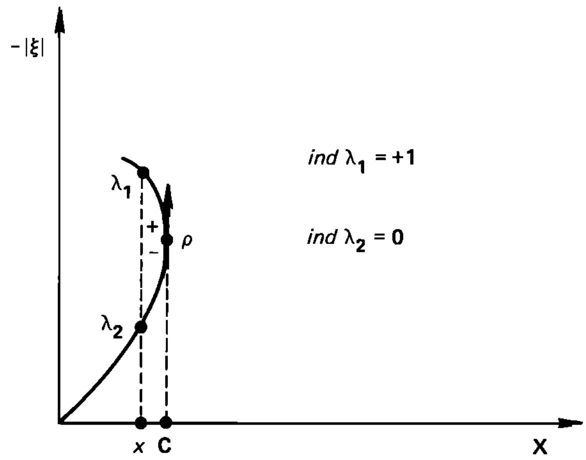

Fig. 10. The $z = z_0$ plane shown in Figure 9.

the z = constant = z_0 cross section of the HMP Λ is shown. The phase space trajectories $H^\tau\lambda_{01}$ and $H^\tau\lambda_{02}$ pass through the points λ_1 and λ_2, which lie in different branches of Λ. Those trajectories project onto GO rays 1 and 2 as displayed. Furthermore, the points λ_1 and λ_2 are regular; their projection on X is x. The projection of the singular point ρ is C; it lies on the caustic of the GO rays. Figure 10 displays the contents of the $z = z_0$ plane in Figure 9. The positive side of the singular point ρ is indicated. Since the trajectory $H^\tau\lambda_{02}$ has not yet crossed Σ upon reaching λ_2, the index of λ_2 is 0. As shown in Figure 9 this corresponds to x being a point on ray 2 before it has passed through the caustic. In contrast, from Figure 10 it is immediately seen that the index of λ_1 is $+1$(ind $l[\lambda_2, \lambda_1] = +1$) and hence, equivalently, that $H^\tau\lambda_{01}$ has crossed Σ, that λ_1 lies on a different branch of Λ than λ_2, and that x lies on ray 1 beyond the caustic. Note that the intersection of two rays at x corresponds to x having two preimages, i.e., that Λ has two branches over x.

3.5. *Half densities*

To complete the phase space GO representation, amplitudes on the LSM Λ and their variations along the phase space trajectories must be defined. This is accomplished with the introduction of densities of order 1/2, half densities for short. The connection between amplitudes defined on Λ and those defined on X and the hybrid spaces Y is provided by the transformation properties of the half densities.

First, consider the more familiar concept of the volume density, density for short. Densities on X and Y have the forms

$$\alpha_X = f(x)\,|dX| \tag{89}$$

$$\alpha_Y = g(y)\,|dY| \tag{90}$$

where $f(x)$ and $g(y)$ are complex-valued functions over X and Y and the (unit) densities

$$dX = dx^1 \cdots dx^n \tag{91}$$

$$dY = dy^1 \cdots dy^n \tag{92}$$

If the transformation μ maps X to Y diffeomorphically,

$$\mu : X \to Y : x \mapsto y = \mu(x) \tag{93}$$

and if

$$J(y/x) = dY/dX \tag{94}$$

is the Jacobian of that transformation, the densities α_X and α_Y are related as

$$\alpha_X = \mu^*\alpha_Y \tag{95}$$

(where μ^* is the pullback through the map μ [*Deschamps*, 1981, Appendix I]) hence their amplitudes (coefficients) are related as

$$f(x) = \mu^*[g(y)\,|J(x/y)|^{-1}] = g(\mu(x))\,|J(y/x)| \tag{96}$$

This represents the well-known "change of variables" result from integration theory.

Similarly, half densities on X and Y are defined to have the forms

$$\beta_X = A(x)\,|dX|^{1/2} \tag{97}$$

$$\beta_Y = B(y)\,|dY|^{1/2} \tag{98}$$

They are related through a diffeomorphism μ as

$$\beta_X = \mu^*\beta_Y \tag{99}$$

and their amplitudes satisfy the transformation rule

$$A(x) = B(\mu(x))\,|J(y/x)|^{1/2} \tag{100}$$

This rule also takes the more symmetric density form

$$A^2(x)\,|dX| = B^2(y)\,|dY| \tag{101}$$

Now we take $\lambda = (\lambda_1, \ldots, \lambda_n)$ to be the coordinates of the LSM Λ and let its unit density be $d\Lambda = d\lambda^1 \cdots d\lambda^n$. (Note that, rigorously, the coordinates λ_i of Λ are related to its phase space coordinates $\lambda_{ps} = (x, \kappa)$ through an (immersion) map $\imath$ as $\lambda_{ps} = \imath(\lambda_i)$. We will not make this distinction explicit because it would add unnecessary complications to the discussion. It is simply a matter of treating Λ itself or as a subset of M. The resultant inaccuracy in all of the following expressions is corrected by replacing any projection map π with $\pi \circ \iota$.) Over a regular set W_r of Λ the amplitude h of the half density on Λ

$$\beta_\Lambda = h(\lambda)\,|d\Lambda|^{1/2} \tag{102}$$

is related through the projection map π_X and the half-density transformation rule to the amplitude of the half density (97) on X as

$$h(\lambda) = A(\pi_X(\lambda))\,|J(x/\lambda)|^{1/2} \tag{103}$$

Similarly, over a set W_s of Λ that is regular with respect to Y, the amplitudes h and B are related as

$$h(\lambda) = B(\pi_Y(\lambda))\,|J(y/\lambda)|^{1/2} \tag{104}$$

Consequently, in an overlap region $W_r \cap W_s$ the amplitudes A and B must satisfy the consistency relation

$$A(x) = B(\mu(x))\,|J(y/x)|^{1/2} \tag{105}$$

where the diffeomorphism

$$\mu = \pi_Y \circ \pi_X^{-1} \tag{106}$$

Equations (103)–(105) provide the desired connections between the GO amplitudes in Λ, X, and Y.

The transport equation for the phase space GO amplitude h in the examples is obtained as follows. The general case is treated briefly in Appendix D. Equation (23), which describes the variation of the divergence factor along the GO rays, has the half-density representation

$$\mathscr{L}_v\,|dX|^{1/2} = \tfrac{1}{2}(\operatorname{div} v)\,|dX|^{1/2} \tag{107}$$

Since

$$\begin{aligned}\mathfrak{L}_v \beta_X &= (\mathfrak{L}_v A)\,|dX|^{1/2} + A\mathfrak{L}_v\,|dX|^{1/2}\\ &= \left[\frac{d}{d\tau}A + \frac{1}{2}(\operatorname{div} v)A\right]|dX|^{1/2}\end{aligned} \tag{108}$$

the half-density form of the transport equation (23) in both examples is

$$\mathfrak{L}_v \beta_X = 0 \tag{109}$$

It is readily shown (see Appendix D) that this is the projection on X of the equation satisfied by β_Λ on Λ:

$$\mathfrak{L}_{\mathscr{H}} \beta_\Lambda = 0 \tag{110}$$

Moreover, the incompressibility of the Hamilton flow gives

$$\mathfrak{L}_{\mathscr{H}}\,|d\Lambda|^{1/2} = 0 \tag{111}$$

Therefore, the amplitude h of β_Λ satisfies the transport equation

$$(d/d\tau)h = 0 \tag{112}$$

This means the phase space GO amplitude is a constant along a trajectory:

$$h(H^\tau \lambda_0) = h(\lambda_0) \tag{113}$$

The initial values $h(\lambda_0)$ on Λ_0 are obtained from those given on X_0 through the transformation rule (103):

$$h(\lambda_0) = A(\pi_X(\lambda_0))\,|J(x_0/\lambda_0)|^{1/2} \tag{114}$$

4. THE FIELD NEAR A CAUSTIC

Maslov recognized the importance of the LSM Λ and the central role it can play in producing global solutions to the GO problem as well as uniform representations in the caustic regions. In his approach the problem is transformed (lifted) from the space X to a problem on a particular LSM Λ where, as noted earlier, the trajectories present no caustics. A GO field compatible with the initial conditions in X is constructed on Λ and transformed back to the space X by means of an operator called the canonical operator. When Λ is regular over X, the canonical operator projects the GO field on Λ to X; the resulting field coincides with the GO field in X. However, when difficulties arise in that projection, which happens when a trajectory crosses the apparent contour Σ of Λ or, correspondingly, when a ray meets a caustic, the canonical operator looks at the GO field on Λ from a different point of view by projecting it on a hybrid space Y. It then generates the desired field in the caustic region as the Fourier transform of that hybrid space GO field. There, uniform approximations are derived readily from this representation.

4.1. *Fourier transform*

Following the notations of *Maslov* [1972] and *Leray* [1972], the Fourier transform

$$\mathscr{F}(\kappa/x): f(x) \mapsto \hat{f}(\kappa) = c\int e^{-\nu\kappa\cdot x} f(x)\,dX \tag{115}$$

where the constant $c = (-\nu/2\pi)^{n/2}$ is precisely defined by taking its argument equal to $(-n\pi/4)$,

$$c = (k/2\pi)^{n/2}\exp(-in\pi/4) \tag{116}$$

and where the integration extends over the entire space X. The Fourier transform $\bar{\mathscr{F}}(x/\kappa)$, which is the inverse of $\mathscr{F}(k/x)$: $\bar{\mathscr{F}} = \mathscr{F}^{-1}$, results from replacing ν

and c in (115) by their complex conjugates, i.e., by $-\nu$ and

$$\bar{c} = (+\nu/2\pi)^{n/2} = (k/2\pi)^{n/2} \exp(in\pi/4) \qquad (117)$$

This definition of the Fourier transform and that of its inverse are only variations of the ordinary one. The inclusion of the factor k in the exponent of the kernel of the transform and the factors $(\pm\nu)^{n/2}$ facilitate asymptotic considerations.

4.2. *Maslov's canonical operator*

The connection between the GO field constructed on Λ and the approximate solution in the space X is provided by the canonical operator $\mathscr{K}_\Lambda$. The operator is a linear mapping of functions defined on an LSM Λ to those on X that has the following characteristics. If λ is a point in a regular set W_r of Λ, it assigns to the GO amplitude h at λ the expression $\mathscr{K}_\Lambda h$ at the point $x = \pi_X(\lambda)$:

$$\mathscr{K}_\Lambda h(x) = e^{iR(x)} h(\pi_X^{-1}(x)) \, | J(\lambda/x) |^{1/2} \qquad (118)$$

where the phase

$$R(x) = k\Phi(x) - (\pi/2) \, ind \, [\pi_X^{-1}(x)] \qquad (119)$$

the generating function of W_r being defined by (53) and (54) as

$$\Phi(x) = \phi(x_0) + \int_{x_0}^{x} \kappa(x) \cdot dx \qquad (120)$$

On the other hand, if λ is a point in a set W_s of Λ that is singular over X but regular over Y, it gives

$$\mathscr{K}_\Lambda h(x) = \bar{\mathscr{F}}(x/y)[e^{iS(y)} h(\pi_Y^{-1}(y)) \, | J(\lambda/y) |^{1/2}] \qquad (121)$$

where $y = \pi_Y(\lambda)$ and the phase

$$S(y) = \Psi(y) - (\pi/2) \, ind \, [\pi_Y^{-1}(y)] \qquad (122)$$

the generating function Ψ of W_s being defined by (67) and (68) as

$$\Psi(y) = \psi(y_0) + \int_{y_0}^{y} (\zeta \cdot dz - x \cdot d\xi) \qquad (123)$$

Note that, as $y = \mu(x)$, the initial point $y_0 = \mu(x_0)$. Furthermore, the initial manifold Λ_0 is also regular over Y, hence the initial phase ψ in Y is the Legendre transform of ϕ:

$$\psi = \mathscr{L}\phi \qquad (124)$$

The minimum number of wave vector coordinates in Y equals the maximum singularity index $N(\lambda)$ for $\lambda \in W_s$. If there are several points λ_l on Λ that project to the same point x in X, and if the λ_l, $l \in r[l \in s]$ are regular (singular) over X, then the canonical operator assigns to the GO amplitude h_l (the restriction of h to a neighborhood of λ_l) the value

$$\mathscr{K}_\Lambda h(x) = \sum_{i \in r} \mathscr{K}_\Lambda h_i(x) + \sum_{j \in s} \mathscr{K}_\Lambda h_j(x) \qquad (125)$$

where each of the terms in the first sum is defined by (118) and the terms of the second sum by (121).

The canonical operator expressions are related readily to GO fields. From (51) and (113) one obtains

$$\begin{aligned} h(\pi_X^{-1}(x)) \, | J(\lambda/x) |^{1/2} &= h(\pi_X^{-1}(x_0)) \, | J(\lambda_0/\lambda) |^{1/2} \, | J(\lambda/x_0) |^{1/2} \\ &= a(x_0) \, | J(x_0/\lambda_0) |^{1/2} \, | J(\lambda_0/x) |^{1/2} \\ &= a(x_0) \, | J(x_0/x) |^{1/2} \end{aligned} \qquad (126)$$

Furthermore, as shown in [*Ziolkowski*, 1980],

$$\mathscr{D}^{-1/2}(x/x_0) = | J(x/x_0) |^{-1/2} \exp\{-i(\pi/2) \, ind \, [\pi_X^{-1}(x)]\} \qquad (127)$$

i.e., the index of the Jacobian matrix $\| (\partial x_i / \partial x_{0j}) \|$ (the number of negative eigenvalues) equals the Morse index. Consequently, at a regular point the canonical operator returns the GO field (27):

$$\mathscr{K}_\Lambda h(x) = e^{\nu\Phi(x)} A(x) = u(x) \qquad (128)$$

Similarly, it can be shown that

$$e^{iS(y)} h(\pi_Y^{-1}(y)) \, | J(\lambda/y) |^{1/2} = e^{\nu\Psi(y)} B(y) = v(y) \qquad (129)$$

where

$$B(y) = \mathscr{D}^{-1/2}(y/y_0) b(y_0) \qquad (130)$$

and

$$b(y_0) = h(\pi_Y^{-1}(y_0)) J^{1/2}(\lambda_0/y_0) = a(\mu(x_0)) J^{1/2}(x_0/y_0) \qquad (131)$$

is the GO field in the mixed space Y. In particular, the operator $\mathscr{Q}(y, D_y)$, defined by the Fourier transform property

$$\mathscr{Q}(y, D_y) = \mathscr{F}(y/x) \circ \mathscr{P}(x, D_x) \circ \bar{\mathscr{F}}(x/y) \qquad (132)$$

and the initial conditions (124) and (131) define the continuation problem in Y corresponding to the one in X. The projection on Y of the phase space trajectories gives the rays in Y, and because the principal symbols of the operators $\mathscr{P}$ and $\mathscr{Q}$ are identical, those rays have the velocity field

$$\frac{d}{d\tau}(\xi, z) = \mathscr{V} = (p_\zeta(y, \Psi_y) - p_x(y, \Psi_y)) \qquad (133)$$

Thus

$$\operatorname{div} \mathscr{V} = \partial_z \cdot [p_\zeta(y, \Psi_y)] + \partial_\xi \cdot [-p_x(y, \Psi_y)] \qquad (134)$$

and

$$\frac{d}{d\tau} B(y) = p_\xi(y, \Psi_y) \cdot \partial_z B(y) - p_x(y, \Psi_y) \cdot \partial_\xi B(y) \tag{135}$$

Furthermore, the projections of the dispersion relation and the amplitude transport equation are

$$p(y, \Psi_y) = 0 \qquad \text{(eikonal)} \tag{136}$$

$$\frac{d}{d\tau} B + \frac{1}{2} (\operatorname{div} \mathscr{V}) B = 0 \qquad \text{(transport)} \tag{137}$$

Consequently,

$$\mathscr{Q}(y, D_y) v(y) = e^{\nu \Psi(y)} \Big\{ p(y, \Psi_y) B(y) + \nu^{-1} \left[\frac{d}{d\tau} B(y) + \frac{1}{2} (\operatorname{div} \mathscr{V}) B(y) \right] + O(\nu^{-2}) \Big\} = 0 \tag{138}$$

Since it also satisfies the initial conditions, $v(y)$ is the GO field in Y. Hence, at a singular point, the canonical operator returns the Fourier transform of the hybrid space GO field

$$\mathscr{K}_\Lambda h(x) = \bar{\mathscr{F}}(x/y)[v(y)] = u(x) \tag{139}$$

Note that the field $u(x)$ is shown immediately with (132) and (138) to be an asymptotic solution of (1):

$$\mathscr{P}(x, D_x) u(x) = \bar{\mathscr{F}}(x/y)[\mathscr{Q}(y, D_y) v(y)] \sim 0 \tag{140}$$

4.3. *The asymptotic Fourier transform* $\mathscr{F}_0$

The Fourier transform $\mathscr{F}(\kappa/x)$ takes the function $f = e^{\nu\Phi} A$, defined over X, to the function $\hat{f}$, defined over the dual space K:

$$\hat{f}(\kappa) = \mathscr{F}(\kappa/x)[f(x)] = c \int e^{\nu[\Phi(x) - \kappa \cdot x]} A(x)\, dX \tag{141}$$

Assume that (for a given κ) the phase $[\Phi(x) - \kappa \cdot x]$ has an isolated nondegenerate stationary point $x_s = s(\kappa)$ (a point x_s is a stationary point with respect to x of the function $\Phi(x) - \kappa \cdot x$ if, for a given κ, $\Phi_x(x_s) = \kappa$; it is also nondegenerate if the Hessian of Φ at x_s is

$$\operatorname{Hess} \Phi(x_s) = \det \| (\partial_{x_i} \partial_{x_j} \Phi)(x_s) \| \neq 0$$

Therefore, in the vicinity of κ the function $s = \Phi_x^{-1}$: $\kappa \mapsto x_s$. The stationary phase approximation of $\hat{f}$ has the form

$$\hat{f}(\kappa) \sim e^{\nu\Psi(\kappa)} B(\kappa) + e^{\nu\Psi(\kappa)} \sum_{j=1}^{\infty} \nu^{-j} B_j(\kappa) \tag{142}$$

where the phase Ψ is the Legendre transform of Φ, $\Psi = \mathscr{L}\Phi$, so that

$$\Psi(\kappa) = \Phi(s(\kappa)) - \kappa \cdot s(\kappa) \tag{143}$$

and, for instance, the amplitude

$$B(\kappa) = A(s(\kappa))\, (dX/dK)^{1/2} \tag{144}$$

so that

$$B^2\, dK = A^2\, dX \tag{144'}$$

The higher-order amplitudes are given, for example, in *Ziolkowski and Deschamps* [1984*a*]. Note that (144′) represents a zeroth-order approximation of Parseval's theorem. When there are no stationary points within the support of A, $\hat{f}$ is asymptotically null: $\hat{f}(\kappa) \sim 0$. When there are several roots, each one contributes to $\hat{f}$ an asymptotic expansion through the same process as when there is only one stationary point. This, of course, assumes the critical points are isolated. If any of the stationary points were "near" to one another, the above process would be inadequate and a uniform expansion would be necessary.

We define the asymptotic Fourier transform (AFT) of range zero ($\mathscr{F}_0(\kappa/x)$) as the operator obtained by truncating the asymptotic series (142) to the term independent of ν, i.e., as the operator taking a function of the form $f = e^{\nu\Phi} A$ over X to the function $F = e^{\nu\Psi} B$ over K, its phase Ψ and amplitude B being defined by (143) and (144), respectively:

$$\mathscr{F}_0(\kappa/x): f(x) = e^{\nu\Phi(x)} A(x) \mapsto F(\kappa) = e^{\nu[\Phi(s(\kappa)) - \kappa \cdot s(\kappa)]} A(s(\kappa))\, (dX/dK)^{1/2} \tag{145}$$

Similarly, with $\bar{\mathscr{F}}(x/\kappa)$ we construct the AFT $\bar{\mathscr{F}}_0(x/\kappa)$ so that

$$\bar{\mathscr{F}}_0(x/\kappa): G(\kappa) = e^{\nu\psi(\kappa)} b(\kappa) \mapsto g(x) = e^{\phi(x)} a(x) \tag{146}$$

where $\phi = \mathscr{L}\psi$, hence

$$\phi(x) = \psi(\sigma(x)) + x \cdot \sigma(x) \tag{147}$$

and

$$a(x) = b(\sigma(x))\, (dK/dX)^{1/2} \tag{148}$$

or

$$a^2\, dX = b^2\, dK \tag{148'}$$

$\kappa_s = \sigma(x)$ being the root of the stationary point condition $\psi_\kappa(\kappa_s) = -x$. The (partial) AFT's $\mathscr{F}_0(y/x)$ and $\bar{\mathscr{F}}_0(x/y)$ are defined analogously—the coordinates unchanged in going from X to Y act as parameters.

The AFT $\mathscr{F}_0$ is a local operator and satisfies properties similar to those satisfied by the exact transform $\mathscr{F}$. For instance, as shown in *Ziolkowski* *and Deschamps* [1984*a*],

$$\bar{\mathcal{F}}_0 = \mathcal{F}_0^{-1} \tag{149}$$

Since $\bar{\mathscr{L}} = \mathscr{L}^{-1}$, hence $\sigma = s^{-1}$, this result is easily confirmed by using (145) and (146). Similar considerations are possible for higher-order terms; the definition of the AFT of range m and its properties are given in *Ziolkowski and Deschamps* [1984*a*]. Its applications in geometrical optics are discussed in *Ziolkowski and Deschamps* [1984*b*]; they also were reported in *Ziolkowski and Deschamps* [1980], and *Ziolkowski* [1980]. Of particular interest is the proof that the GO solutions along the rays of corresponding continuation problems in two hybrid spaces are related by the AFT. This means

$$v(y) = \mathcal{F}_0(y/x)[u(x)] \tag{150}$$

Consequently, Maslov's solution to the continuation of a field through a caustic region has the alternate compact representation:

$$u(x) = \begin{cases} u(x) & (151a) \\ \quad \text{if } x \text{ is away from any caustic} & \\ \{\bar{\mathcal{F}}(x/y) \circ \mathcal{F}_0(y/x)\}[u(x)] & (151b) \\ \quad \text{if } x \text{ is near a caustic} & \end{cases}$$

which was first introduced by *Ziolkowski and Deschamps* [1980]. The AFT operator $\mathcal{F}_0$ effectively cancels the singularities in the GO field $u(x)$. Clearly, if x is sufficiently far from a caustic, the operator $\bar{\mathcal{F}}$ can be replaced with $\bar{\mathcal{F}}_0$ in (151*b*), and (149) returns (151*a*) immediately, i.e., away from a caustic, (151*b*) reduces to the GO field $u(x)$.

4.4. *Uniform asymptotics*

All of the above representations of the field at a point x near a caustic share a common "oscillatory integral" form [*Hörmander*, 1971; *Duistermäat*, 1974]:

$$u(x, z) = \int e^{\nu\Gamma(\xi,z;x)} g(\xi, z)\, d\xi \tag{152}$$

Away from a caustic, the stationary phase approximation of this integral recovers the GO field. However, that approximation breaks down near a caustic because the stationary points of Γ (with respect to ξ) coalesce, i.e., are degenerate there. The caustic points are the singular points of the stationary phase mapping σ. Equivalently, the stationary point set of Γ (those x, z, ξ such that $\Gamma_\xi = 0$) generates the LSM Λ, and the degenerate critical point set of Γ coincides with the singular set Σ.

Uniform asymptotic approximations to the oscillatory integral (152) can be obtained. The general representation (152) is transformed to a canonical integral corresponding to the specific type of degenerate critical point, hence caustic, involved. Each caustic can be characterized as an elementary caustic or a combination of elementary caustics. Uniform expansions of the corresponding canonical integrals are known. For instance, the Airy function is characteristic of the fold caustic that occurs in the LLP. Similarly, the parabolic cylinder function correctly describes the field behavior near the cusp point of the caustic in the HMP. The fold and cusp caustics are the only elementary caustics in two dimensions; any other two-dimensional caustic can be decomposed into fold and cusp components. The general scheme that describes the correspondence between caustics, singularities of mappings, their unfoldings, oscillatory integrals, and their resultant uniform expansions has been discussed, for instance, by *Arnol'd* [1968, 1972, 1974] and by *Duistermäat* [1974]. These concepts also are central to catastrophe theory [*Poston and Stewart*, 1978], a subject beyond the scope of this paper.

These results can be interpreted in another way. If the structure of the caustic is known, one can choose locally the canonical integral form of (152). The solution can be generated by matching the asymptotic expansion of the integral away from the caustic to its GO values. This is the essence of the "Relevant Function Method" advocated by *Ludwig* [1966] and *Kravtsov* [1964*a*, *b*]. In comparison, the Maslov method has the advantage that the solution is generated in an integral form that is valid in any region of the caustic and that can be transformed locally into this relevant function form.

4.5. *Linear layer problem*

According to Appendix A, the GO field along each of the parabolic rays in the linear layer problem has the form

$$u(x, z) = \exp\{\nu[\Phi_c(x) - (2/3\alpha)\zeta^3(z)]\}(\zeta_0/\zeta(z))^{1/2} \tag{153}$$

where the phase at the caustic $z = \mathscr{Z} = \zeta_0^2/\alpha$ is

$$\Phi_c(x) = \xi_0 x + (2/3\alpha)\zeta_0^3 \tag{154}$$

Since two rays intersect each point in the lit region, the total GO field is

$$u(x) = e^{\nu\Phi_-(x)}|\zeta_0/\zeta|^{1/2} + e^{\nu\Phi_+(x)}e^{-i\pi/2}|\zeta_0/\zeta|^{1/2} \tag{155}$$

where the phases

$$\Phi_{\pm}(x) = \Phi_c(x) \pm (2\alpha^{1/2}/3)(\mathscr{X} - z)^{3/2} \tag{156}$$

and the term

$$|\zeta| = |\alpha|^{1/2}|\mathscr{X} - z|^{1/2} \tag{157}$$

The second term represents the field along the ray descending from the caustic and contains the $(\pi/2)$ phase shift characteristic of the passage through a fold caustic.

The solutions to Hamilton's equations in Appendix A indicate that the hybrid space Y with coordinates (x, ζ) is free of caustics. In fact, the rays in Y

$$\begin{cases} x = x_0 + \xi_0 \tau \\ \zeta = \zeta_0 - (\alpha/2)\tau \end{cases} \tag{158}$$

as shown in Figure 11, are all parallel. The relationship of these rays to those in X and to the phase space trajectories is illustrated in Figure 12. The hybrid space differential operator

$$\mathscr{Q}(y, D_y) = \bar{\mathscr{F}}(y/x) \circ \mathscr{P}(x, D_x) \circ \mathscr{F}(x/y)$$
$$= \tfrac{1}{2}\{D_x^2 - \alpha D_\zeta - (1 - \zeta^2)\} \tag{159}$$

yields the GO equations

$$\Psi_x^2 - \alpha\Psi_\zeta - (1 - \zeta^2) = 0 \quad \text{(eikonal)} \tag{160a}$$

$$\frac{d}{d\tau} B + \frac{1}{2}\Psi_{xx} B = 0 \quad \text{(transport)} \tag{160b}$$

Thus the hybrid space GO field is

$$v(y) = \exp\{v[\Phi_c(x) - \mathscr{X}\zeta + \zeta^3/3\alpha]\}(-2\zeta_0/\alpha)^{1/2} \tag{161}$$

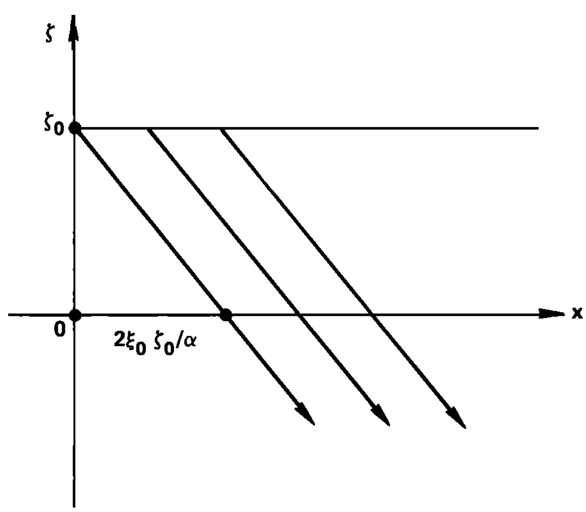

Fig. 11. The hybrid space rays in the linear layer problem.

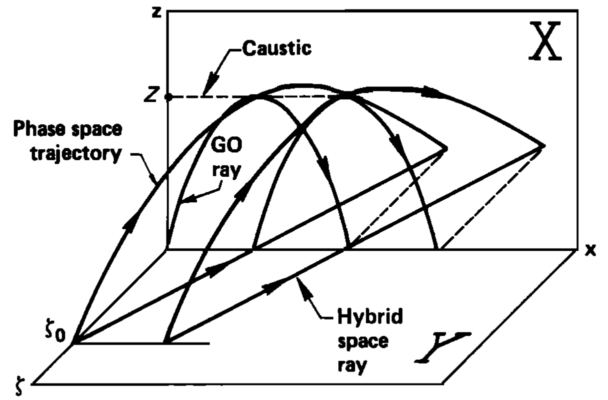

Fig. 12. The hybrid space rays in the homogeneous medium problem.

Consequently, the field near the caustic is

$$u(x) = \bar{\mathscr{F}}(x/y)[v(y)]$$
$$= \bar{c}e^{v\Phi_c(x)}(-2\zeta_0/\alpha)^{1/2} \int d\zeta e^{v\zeta z} \cdot \{e^{-v[\zeta\mathscr{X} - \zeta^3/3\alpha]}\}$$
$$= e^{-i\pi/4}(4\pi\zeta_0)^{1/2}(k/\alpha)^{1/6}e^{v\Phi_c(x)} \cdot Ai((\alpha/k)^{1/3}(\mathscr{X} - z)) \tag{162}$$

where the Airy function

$$Ai(w) = \int \exp[i(wt - t^3/3)]\, dt/2\pi \tag{163}$$

The Airy function behavior characteristic of a fold caustic has been obtained simply and naturally with Maslov's method. Note that the $k^{1/6}$ dependence, also characteristic of the increased field intensity near a fold caustic, has been obtained naturally too.

The satisfaction of (150) is readily verified. The root $z = s_x(\zeta)$ of $\Phi_z(x, s_x(\zeta)) = \zeta$ is $z = (\zeta_0^2 - \zeta^2)/\alpha$. Thus the transformations of the phase

$$\Phi(x, s_x(\zeta)) - \zeta \cdot s_x(\zeta) = \phi(x_0)$$
$$+ \xi_0(x - x_0) + (2/3\alpha)(\zeta_0^3 - \zeta^3) - \zeta(\zeta_0^2 - \zeta^2)/\alpha$$
$$= \xi_0 x - \mathscr{X}\zeta + \zeta^3/3\alpha \equiv \Psi(x, \zeta) \tag{164a}$$

and the amplitude

$$A(x, s_x(\zeta))\,(dz/d\zeta)^{1/2} = (\zeta_0/\zeta)^{1/2}(-2\zeta/\alpha)^{1/2}$$
$$= (-2\zeta_0/\alpha)^{1/2} = B(x, \zeta) \tag{164b}$$

yield

$$\mathscr{F}_0(y/x)[u(x)] = v(y) \tag{164c}$$

Clearly, the representation $\{\bar{\mathscr{F}}(x/y) \circ \mathscr{F}_0(y/x)\}u(x)$ also reduces to (162).

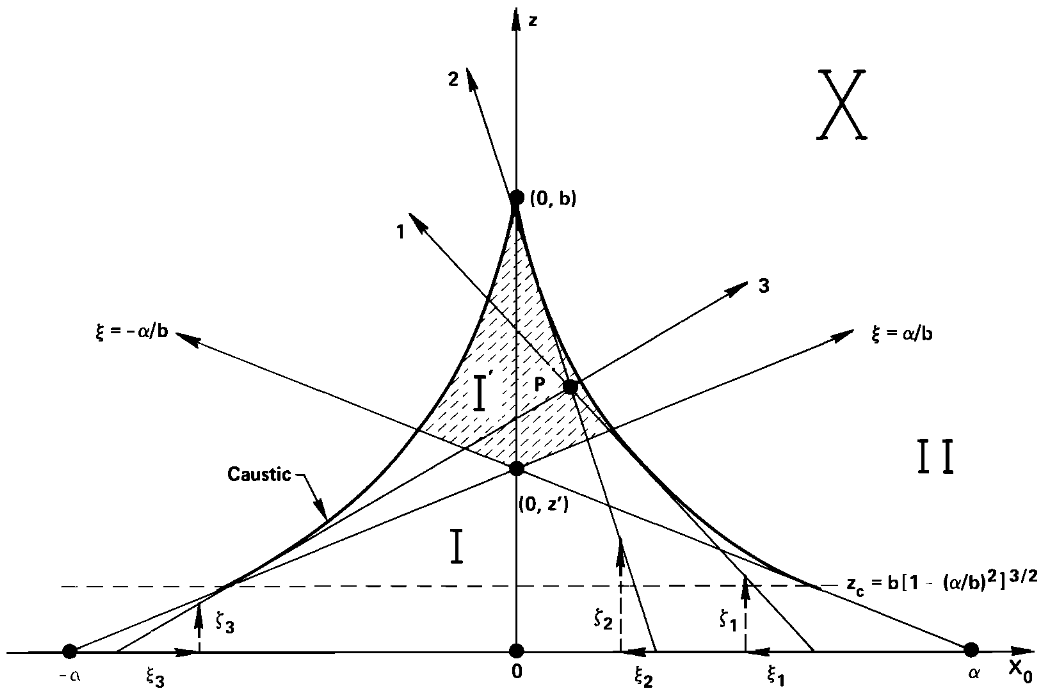

Fig. 13. Ray system generated by the initial phase $\phi(x_0) = -x_0^2/2b$ and its envelope—a cusp caustic.

4.6. *Homogeneous medium problem*

According to Appendix A, the GO field along a ray in the homogeneous medium problem has the form

$$u(x) = \exp\,[\nu(\tau - x_0^2/2b)]\left[\frac{b^2 - x_0^2}{b^2 - x_0^2 - b\tau}\right]^{1/2} a(x_0) \tag{165}$$

The ray system and its envelope (the cusp caustic) are illustrated in detail in Figure 13. The region enclosed by the support of the initial field ($|x_0| \le \alpha$), the caustic, and the line segments from the endpoints ($\pm\alpha$, 0) to the caustic points ($\pm x_c$, z_c) where $x_c = \pm\alpha^3/b^2$ and $z_c = b(1 - (\alpha/b)^2)^{3/2}$ is denoted region I. The subset of region I bounded by the caustic and the rays with direction cosines $\xi = \pm\alpha/b$ intersecting at $(0, z')$, $z' = (b^2 - \alpha^2)^{1/2}$, is denoted region I′. In region I, for $0 < z < z_c$ there is only one ray through each point. In contrast there are two rays through each point in region I with $z > z'$ but not in region I′, and there are three rays through each point in region I′. As shown, the point P in region I′ is the intersection point of rays 1, 2, and 3. The wave vector (ξ, ζ) of each of these rays is also indicated. Notice that both rays 1 and 3 have already encountered the caustic; ray 2 has not. If P were near the positive x fold portion of the caustic, rays 1 and 2 would nearly coincide. At most, one real ray reaches any point in region II, the complement of region I. The rays with $\xi = \pm\alpha/b$ are the shadow boundaries in region II.

The corresponding level curves of Λ in the (x, ξ) plane at constant z values are shown in Figure 14. The intersection of the x = constant planes with those level curves also are given; they will be called "projection lines" for short. Since $z = \zeta\tau$, these level curves represent the curves $\Lambda_\tau = H^\tau\Lambda_0$. The $z = 0$ curve in Figure 14*a* is Λ_0. The projection lines indicate that every point of Λ_0 is regular over X and the hybrid space Y with coordinates (ξ, z). The curves in Figures 14*b*, *c*, and *d* are representative of the level curves for $0 < z < z_c$, $z_c < z < z'$, and $z' < z < b$, respectively. The projection lines in Figure 14*b* show that there is only one ray through a point in the corresponding portion of region I. In Figure 14*c* the projection lines indicate, for example, that two rays intersect at point P. Furthermore, the projection line through the caustic point C intersects Λ at the corresponding singular point. The connection with Figures 9 and 10 is immediate. The projection lines in Figure 14*d* indicate the intersection of three rays through a point in region I′ and the passage of only one ray through a point in region II. Note that the coincidence of the two rays at the caustic is clearly

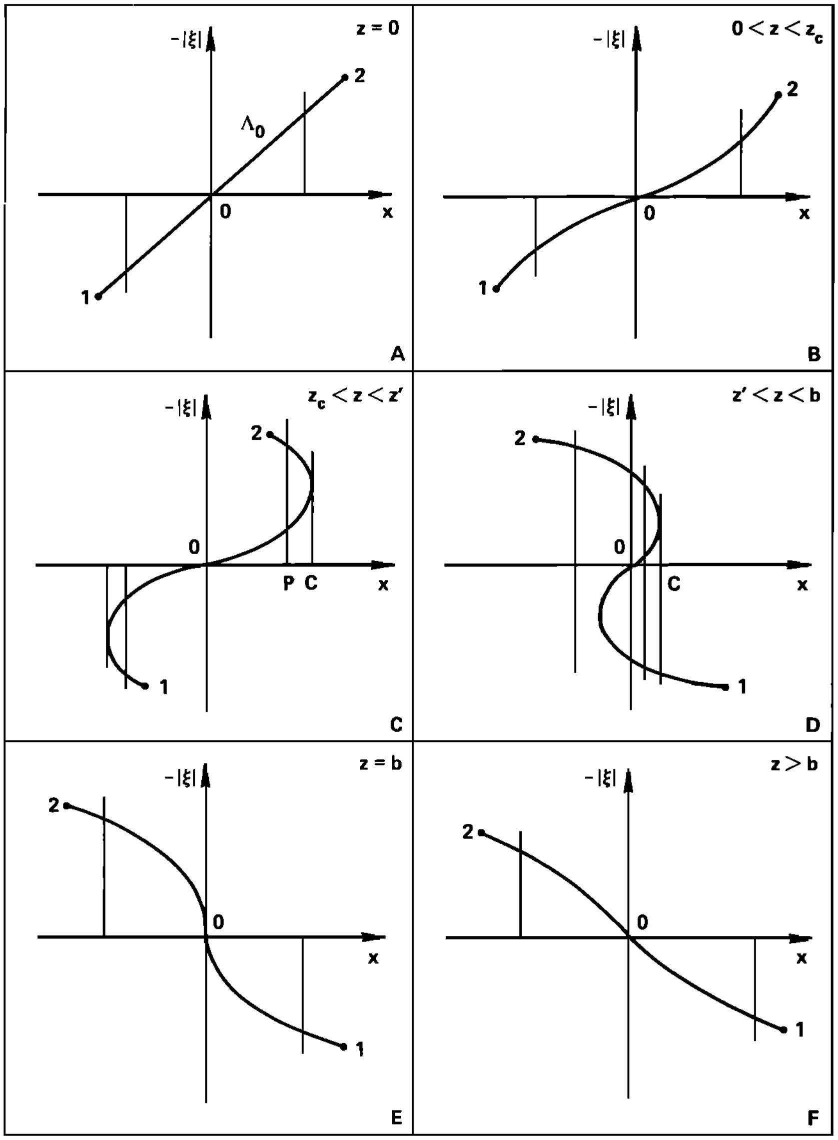

Fig. 14. Level curves of the Lagrangian $\Lambda = \{x + b\xi - [\xi/\zeta(\xi)]z\}$.

indicated by Figures 14c and d. The curve in Figure 14e represents the level curve through the cusp point $(0, b)$. A comparison with Figure 14d indicates the occurrence of the coincidence of three rays at the cusp point. Every other point is regular over X. Figure 14f shows that only one ray reaches a point in region II. Note that in each figure every point of Λ is regular with respect to Y. Also note that in each figure the endpoints of the level curves, which lie on the rays with $\xi = \alpha/b$ and $\xi = -\alpha/b$, are labeled by 1 and 2, respectively.

According to Appendix A, the rays in the hybrid space Y are defined as

$$\begin{cases} \xi = \xi_0 \\ z = \zeta(\xi)\tau \end{cases} \tag{166}$$

This means, as shown in Figure 15, that the rays in Y are parallel. The relationship of these rays to those in X and to the phase space trajectories is illustrated in Figure 9. The hybrid space differential operator

$$\mathcal{Q}(y, D_y) = \tfrac{1}{2}\{D_z^2 - (1 - \xi^2)\} \tag{167}$$

yields the GO equations

$$\Psi_z^2 - (1 - \xi^2) = \Psi_z^2 - \zeta^2(\xi) = 0 \qquad \text{(eikonal)} \tag{168a}$$

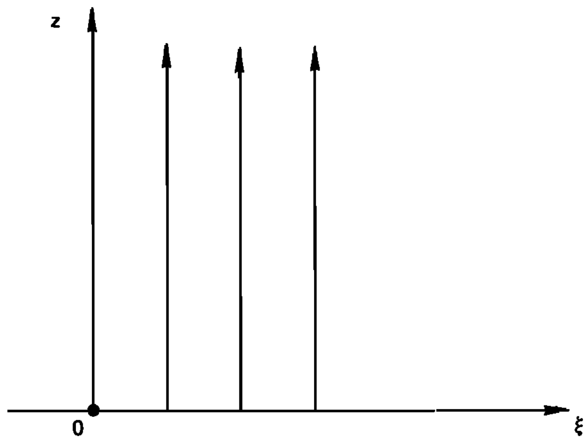

Fig. 15. Relationships between the phase space trajectories and the rays in X and Y.

$$\frac{d}{d\tau} B + \frac{1}{2} \Psi_{zz} B = 0 \qquad \text{(transport)} \tag{168b}$$

Thus the hybrid space GO field

$$v(\xi, z) = e^{\nu\Psi(\xi,z)}B(\xi, z) = e^{\nu\zeta(\xi)z}v(\xi, 0) \tag{168c}$$

consists of a phase propagation term $\exp(\nu\zeta z)$ and the initial field

$$\begin{aligned} v(\xi, 0) &= v(\xi_0, 0) = \mathcal{F}_0(\xi_0/x_0)[u(x_0, 0)] \\ &= e^{\nu b\xi^2/2}e^{-i\pi/2}b^{1/2}a(-b\xi) = e^{\nu\psi(\xi)}B(\xi, 0) \end{aligned} \tag{168d}$$

Equation (150) is verified readily. Because ξ is a constant, the root $s_z(\xi)$ of the stationary phase relation $\Phi_x(s_z(\xi), z) = \xi$ coincides with its value at $z = 0$: $s_z(\xi) = s_0(\xi) = -b\xi$. Moreover,

$$\frac{dx}{d\xi} = (z/\zeta^3) - b \tag{169}$$

Therefore, the transformations of the phase

$$\Phi(s_z(\xi), z) - \xi x = [\phi(-b\xi) - \xi x_0] + \zeta(\xi)z \equiv \Psi(\xi, z) \tag{170a}$$

and the amplitude

$$\begin{aligned} &A(s_z(\xi), z)\,(dx/d\xi)^{1/2} \\ &\quad = a(-b\xi)[1 - (z/b\zeta^3)]^{-1/2}[(z/\zeta^3) - b]^{1/2} \\ &\quad = e^{-i\pi/2}b^{1/2}a(-b\xi) = B(\xi, 0) \equiv B(\xi, z) \end{aligned} \tag{170b}$$

yield

$$v(\xi, z) = \mathcal{F}_0(\xi/x)[u(x, z)] \tag{170c}$$

or equivalently

$$v(\mathscr{y}) = \mathcal{F}_0(\mathscr{y}/\mathscr{x})[u(\mathscr{x})] \tag{170d}$$

Since there are no caustics in Y, the approximate solution has everywhere the representation

$$\begin{aligned} u(\mathscr{x}) &= \bar{\mathcal{F}}(\mathscr{x}/\mathscr{y})[v(\mathscr{y})] = \bar{c} \int d\xi\; e^{\nu\xi x}[e^{\nu\zeta(\xi)z}v(\xi, 0)] \\ u(\mathscr{x}) &= \int d\xi\; e^{\nu\Gamma(\mathscr{x},\xi)}g(\xi) \end{aligned} \tag{171a}$$

where

$$\Gamma(\mathscr{x}, \xi) = (b\xi^2/2) + \zeta(\xi)z + \xi x \tag{171b}$$

and

$$g(\xi) = (2\pi b/k)^{1/2}a(-b\xi) \tag{171c}$$

It is readily seen that the stationary points of Γ generate Λ,

$$\begin{aligned} &\{(x, z, \xi) \mid \Gamma_\xi(x, z, \xi) = b\xi + x - (\xi/\zeta)z \\ &\quad = \gamma(x, z, \xi) = 0\} \equiv \Lambda \end{aligned} \tag{172}$$

and the degenerate stationary points coincide with Σ:

$$\{(x, z, \xi) \mid \Gamma_{\xi\xi}(x, z, \xi) = b - z/\zeta^3 = \gamma_\xi(x, z, \xi) = 0\} \equiv \Sigma \tag{173}$$

Note that at the cusp point $(0, b)$ one has $\Gamma_\xi = \Gamma_{\xi\xi} = \Gamma_{\xi\xi\xi}(0, b, \xi) = 0$, where $\Gamma_{\xi\xi\xi}(x, z, \xi) = -3\xi z/\zeta^5$.

Away from the caustic, the stationary phase approximation of the integral (171a) recovers the GO field. For instance, in region I′ the GO field at the intersection of rays 1, 2, and 3 is

$$u(\mathscr{x}) = \sum_{j=1}^{3} \exp[\nu\Gamma(\mathscr{x}, \xi_j) + i\pi/2]g(\xi_j)\Gamma_{\xi\xi}^{-1/2}(\mathscr{x}, \xi_j) \tag{174}$$

where $\xi_j = \xi_{0j}/b$ so that

$$\begin{aligned} \Gamma(\mathscr{x}, \xi_j) &= (b\xi_j^2/2) + \zeta_j z + \xi_j x \\ &= (-x_{0j}^2/2b) + \zeta_j z + \xi_j(x - x_0) \\ &= \phi(x_{0j}) + \tau_j \end{aligned} \tag{175a}$$

and

$$\begin{aligned} e^{i\pi/2}g(\xi_j)\Gamma_{\xi\xi}^{-1/2}(\mathscr{x}, \xi_j) &= b^{1/2}a(x_{0j})(b - z/\zeta_j^3)^{-1/2} \\ &= a(x_{0j})\left[\frac{b^2 - x_{0j}^2}{b^2 - x_{0j}^2 - b\tau_j}\right]^{1/2} \end{aligned} \tag{175b}$$

The divergence factor

$$\left[\frac{b^2 - x_{0j}^2}{b^2 - x_{0j}^2 - b\tau_j}\right]^{1/2} = \left|\frac{b^2 - x_{0j}^2}{b^2 - x_{0j}^2 - b\tau_j}\right|^{1/2} \cdot \exp(-i(\pi/2)\varepsilon_j) \tag{176a}$$

where

$$\varepsilon_j = \begin{cases} 0 & j = 2 \\ 1 & j = 1, 3 \end{cases} \tag{176b}$$

The additional phase factor $e^{-i(\pi/2)}$ for rays 1 and 3 indicates they have passed through the caustic as shown in Figure 13. This also is confirmed with the index calculation by using Figure 14*d*.

Near the caustic, the integral (171*a*) precisely defines the approximate solution and could be calculated analytically or numerically. Its uniform expansion, however, is usually adequate. Near the fold part of the caustic (away from the cusp point), only two stationary points coalesce as indicated above. Consider the case where rays 1 and 2 are nearly coincident. The integral (171*a*) is transformed into the canonical form

$$I(x, \nu) = \int E(\eta) \exp [\nu F(x, \eta)]\, d\eta \tag{177}$$

where the phase

$$F(x, \eta) = \rho_0 + \rho_1 \eta - \eta^3/3 \tag{178a}$$

the parameters

$$\rho_0 = [\Gamma(x, \xi_1) + \Gamma(x, \xi_2)]/2 \tag{178b}$$

$$\rho_1 = \{(3/4)[\Gamma(x, \xi_1) - \Gamma(x, \xi_2)]\}^{1/2} \tag{178c}$$

and where the amplitude

$$E(\eta) = \bar{c}g(\xi(\eta))J(\xi/\eta) \tag{179}$$

Using the results in *Duistermäat* [1974], the leading term of the asymptotic expansion of (177) is

$$I_0(x, \nu) = (2\pi b)^{1/2} k^{1/6} e^{-i\pi/4} e^{\nu\rho_0} \left[\frac{2}{-\Gamma_{\xi\xi\xi}(x, \xi_{00})}\right]^{1/3} \cdot a(x_{00}) Ai(k^{2/3}\rho) \tag{180}$$

where $\xi_{00} = (\xi_1 + \xi_2)/2 = -(x_{01} + x_{02})/2b = -x_{00}/b$. The Airy function and $k^{1/6}$ dependence appear as expected. Accounting for the contribution from the field along ray 3, the field near the positive x fold of the caustic in region I′ is

$$u(x) = u(x_{03}, \tau_3) + I_0(x, \nu) \tag{181}$$

Finally, near the cusp point, all three stationary points coalesce, i.e., $\Gamma_\xi = \Gamma_{\xi\xi} = \Gamma_{\xi\xi\xi} = 0$ there. The integral (171*a*) is transformed into the canonical form (177), but now with the phase

$$F(x, \eta) = \rho_0 + \rho_1 \eta + \tfrac{1}{2}\rho_2 \eta^2 + \tfrac{1}{4}\eta^4 \tag{182a}$$

Accounting for the symmetry about the z axis, $\rho_1 = 0$, $\rho_0 = \Gamma(x, 0)$, and

$$\rho_2 = 2[\Gamma(x, 0) - \Gamma(x, \xi_1)]^{1/2} \tag{182b}$$

so that

$$\rho_0 - \rho_2^2/8 = [\Gamma(x, 0) + \Gamma(x, \xi_1)]/2 \tag{182c}$$

The leading term of the asymptotic expansion along the z axis is then

$$u(0, z) = e^{-i\pi/8}(b/2)^{1/2} k^{1/4} \exp [\nu(\rho_0^2 - \rho_2^2/8)] \cdot \left[\frac{3}{-\partial_\xi^4 \Gamma(0, z, 0)}\right]^{1/4} a(0)\mathfrak{D}_{-1/2}(e^{-i\pi/4}(k/2)^{1/2}\rho_2) \tag{183}$$

where the parabolic cylinder function ($Re\ \mu > 0$)

$$\mathfrak{D}_{-\mu}(w) = \frac{2e^{x^2/4}}{\Gamma(\mu)} \int_0^\infty t^{2\mu - 1} e^{-(w+t^2)^2/2}\, dt \tag{184}$$

Note that the field is described by the parabolic cylinder function and shows a $k^{1/4}$ increase in magnitude. Both properties are characteristic of the behavior of a field near a cusp point.

5. SUMMARY

The GO method provides a simple and physically appealing solution of a continuation problem. It generates an asymptotic approximation based on local properties of the system, hence its attractiveness. However, it fails near a caustic.

The GO approach has a phase space representation. The principal symbol of the differential operator of the problem is a function over the phase space M. Its kernel, the hypersurface $\mathfrak{S}$ in M, defines the dispersion relation. The principal symbol also acts as the Hamiltonian for the phase space trajectories. The (Hamilton) flow associated with the velocity field of those trajectories is symplectic, volume preserving, incompressible, and it lies in $\mathfrak{S}$. The phase space continuation is based over that flow.

The phase of the initial field picks out a particular set of the phase space trajectories, those with initial points in the submanifold Λ_0 generated by ϕ. The union of those trajectories is a Lagrangian submanifold of M(LSM) Λ. Not every point of Λ is regular with respect to the original problem space X. The projection to X of these singular points coincides with the caustic. On the other hand there is at least one hybrid space Y over which a singular point is regular. The LSM Λ can be generated with a set of phase functions defined over the hybrid spaces. Two phase functions that generate overlapping subsets of Λ are related by a Legendre transformation over the overlap region. Furthermore, an index defined by the intersection number of a trajectory with the singular set Σ of Λ describes the $(\pi/2)$ phase shifts associated with the continuation past a caustic.

The phase space GO amplitude satisfies a transport equation along the trajectories composing Λ. Its initial conditions are generated from a half-density transformation of the initial amplitude. The half-density transformation rule also connects the amplitude on Λ with those in the hybrid spaces X and Y. In contrast with the rays in any hybrid space the trajectories in phase space never form a caustic, hence the amplitude transport on Λ is defined globally.

Maslov's canonical operator defines the approximate solution to a problem by transforming the phase space GO solution to X. Over regular points of Λ it projects that GO field directly to X; over singular points it Fourier transforms to X the projection of that GO field on a hybrid space Y. The former recovers the GO field $u(\mathscr{x})$ in X. The latter returns an integral expression from which uniform expansions can be derived.

It was shown that the projection of the phase space GO field on Y coincides with the hybrid space GO field $v(\mathscr{y})$. Thus Maslov's expression for the field near a caustic can be represented as $\bar{\mathscr{F}}(\mathscr{x}/\mathscr{y})[v(\mathscr{y})]$. Moreover, the GO solution in Y is related to the GO solution in X by the asymptotic Fourier transform (of range zero) $\mathscr{F}_0$; hence the caustic region solution also can be represented directly in terms of the GO field in X as $\{\bar{\mathscr{F}}(\mathscr{x}/\mathscr{y})\circ\mathscr{F}_0(\mathscr{y}/\mathscr{x})\}u(\mathscr{x})$. Maslov's method and these representations were illustrated with two examples: plane wave continuation through a fold caustic in a linear layer medium and continuation through a cusp caustic in a homogeneous medium of a field with an initial quadratic phase.

The representation of a field near a caustic generated with Maslov's method has some drawbacks aside from its mathematical complexity. The solution is not defined over the shadow region because the LSM Λ is not: real rays reach only lit region points. Expressions for higher-order asymptotic terms are not available. Contributions to the approximate results from points on the boundaries of the problem are not taken into account. However, extensions of the alternate hybrid space representations overcome these deficiencies. In particular, higher-order terms can be incorporated into the caustic region solution simply as $\{\bar{\mathscr{F}}(\mathscr{x}/\mathscr{y})\circ\mathscr{F}_m(\mathscr{y}/\mathscr{x})\}u(\mathscr{x})$, and the resultant expressions can be continued analytically into the shadow region (the stationary points become (complex) saddle points corresponding to complex rays). This hybrid space approach was developed in *Ziolkowski* [1980]; a paper summarizing those results is currently in preparation. Nonetheless, Maslov's approach provides a systematic method for obtaining approximate solutions that incorporates a great deal of physical insight into the GO continuation process.

APPENDIX A. GEOMETRICAL OPTICS FIELDS IN THE LINEAR LAYER (LLP) AND THE HOMOGENEOUS MEDIUM (HMP) PROBLEMS

Dispersion relation

$$\tfrac{1}{2}[\xi^2 + \zeta^2 - (1 - \alpha z)] = 0 \qquad \text{(LLP)}$$

$$\tfrac{1}{2}[\xi^2 + \zeta^2 - 1] = 0 \qquad \text{(HMP)}$$

Hamilton's equations

$$d_\tau(x, z, \xi, \zeta) = (\xi, \zeta, 0, -\alpha/2) \qquad \text{(LLP)}$$

$$d_\tau(x, z, \xi, \zeta) = (\xi, \zeta, 0, 0) \qquad \text{(HMP)}$$

Ray equations

$$(x, z, \xi, \zeta) = \left(x_0 + \xi\tau, \zeta_0\tau - \frac{\alpha}{4}\tau^2, \xi_0, \zeta_0 - \frac{\alpha\tau}{2}\right) \qquad \text{(LLP)}$$

$$(x, z, \xi, \zeta) = (x_0 + \xi\tau, \zeta\tau, \xi_0, \zeta_0) \qquad \text{(HMP)}$$

Phase continuation

$$\begin{aligned}\Phi(\mathscr{x}) &= \phi(x_0) + \tau\{1 - (\alpha\tau/2)[\zeta_0 - (\alpha\tau/6)]\} \\ &= \xi_0 x + (2/3\alpha)[\zeta_0^3 \mp \zeta_0^2 - \alpha z]^{3/2}\end{aligned} \qquad \text{(LLP)}$$

The upper (lower) sign is associated with a ray ascending toward (descending from) the caustic.

$$\Phi(\mathscr{x}) = \phi(x_0) + \tau = -(x_0^2/2b) + [(x - x_0)^2 + z^2]^{1/2} \qquad \text{(HMP)}$$

Jacobian

$$\begin{aligned}J(x_0, \tau) &= \zeta(1 + \phi_2\tau) + (\xi^2\phi_2/\zeta_0)\tau \\ &= \zeta = \zeta_0 - (\alpha\tau/2)\end{aligned} \qquad \text{(LLP)}$$

$$J(x_0, \tau) = \zeta[1 + (\tau\phi_2/\zeta^2)] = (\zeta/b)[b - (\tau/\zeta^2)] \qquad \text{(HMP)}$$

The factor $\phi_2 = \partial_{x_0}^2\phi = d\xi_0/dx_0$.

Amplitude

$$A(x_0, \tau) = [1 - (\alpha/2\zeta_0)\tau]^{-1/2} = (\zeta_0/\zeta)^{1/2} \qquad \text{(LLP)}$$

$$A(x_0, \tau) = b^{1/2}[b - (\tau/\zeta^2)]^{-1/2}a(x_0) \qquad \text{(HMP)}$$

APPENDIX B. GEOMETRICAL OPTICS SOLUTION TO A GENERAL PROBLEM

Consider the continuation problem defined by the differential equation (1) and the initial conditions (5). The operator $\mathscr{P}(\mathscr{x}, D_\mathscr{x})$ is assumed to be a linear partial differential operator with smooth coefficients, a well-defined ordering of its operations, and the asymptotic characterization represented by (34):

$$e^{-\nu\kappa\cdot x}\mathscr{P}(x, D_x)e^{\nu\kappa\cdot x} = p(x, \kappa) + \nu^{-1}p_1(x, \kappa) + \mathcal{O}(\nu^{-2}) \tag{A1}$$

It is readily shown that [*Ziolkowski*, 1980; *Leray*, 1972]

$$\mathscr{P}(x, D_x)[e^{\nu\Phi(x)}A(x)] = e^{\nu\Phi(x)}\{p(x, \kappa)A(x) + \nu^{-1}[p_\kappa(x, \Phi_x)\cdot A_x(x) + c(x, \Phi_x)A(x)] + \mathcal{O}(\nu^{-2})\} \tag{A2}$$

where

$$\begin{aligned} c(x, \Phi_x) &= p_1(x, \Phi_x) + \tfrac{1}{2}\sum_{j,l=1}^{N} p_{\kappa_j\kappa_l}(x, \Phi_x)\Phi_{x_jx_l} \\ &= c_0(x, \Phi_x) + \tfrac{1}{2}\partial_x\cdot[p_\kappa(x, \Phi_x)] \end{aligned} \tag{A3}$$

$$c_0(x, \kappa) = p_1(x, \kappa) - \tfrac{1}{2}\sum_{j=1}^{n} p_{x_j\kappa_j}(x, \kappa) \tag{A4}$$

The reason for this appendix is to have the reader recognize that the term $c_0(x, \kappa)$ is present in the general case. It vanishes identically in the LLP and the HMP. This term is well defined over the phase space and is called the subprincipal symbol of the operator $\mathscr{P}(x, D_x)$.

As in the LLP and HMP the eikonal equation is

$$p(x, \Phi_x) = 0 \tag{A5}$$

and the rays are defined by Hamilton's equations

$$\begin{cases} \dfrac{dx}{d\tau} = p_\kappa(x, \Phi_x) \\ \dfrac{d\kappa}{d\tau} = -p_x(x, \Phi_x) \end{cases} \tag{A6}$$

The phase continuation along a ray that connects x_0 to x is governed by the equation

$$\frac{d}{d\tau}\Phi = \kappa\cdot\frac{dx}{d\tau} = \kappa\cdot p_\kappa(x, \Phi_x) \tag{A7a}$$

which has the solution

$$\Phi(x) - \phi(x_0) = \int_{x_0}^{x}\kappa\cdot dx = \int_0^\tau(\kappa\cdot p_\kappa)\, d\tau \tag{A7b}$$

The transport equation

$$p_\kappa(x, \Phi_x)\cdot A_x(x) + c(x, \Phi_x)A(x) = 0 \tag{A8a}$$

is reduced by the ray equations to the ordinary differential equation

$$\frac{d}{d\tau}A + c(x, \Phi_x)A = 0 \tag{A8b}$$

Because the Jacobian of the ray coordinates (α, τ) to the space coordinates x satisfies (23),

$$\frac{d}{d\tau}[\ln J] = \partial_x\cdot[p_\kappa(x, \Phi_x)] \tag{A9}$$

the amplitude $\tilde{A} = J^{1/2}A$ satisfies

$$\frac{d}{d\tau}\tilde{A} + c_0(x, \Phi_x)\tilde{A} = 0 \tag{A10}$$

Consequently, if

$$\mathscr{D}(x/x_0) = [J(\alpha, \tau)/J(\alpha, 0)]\exp\left[2\int_0^\tau c_0(x, \Phi_x)\, d\tau\right] \tag{A11}$$

the amplitude solution is

$$A(x) = \mathscr{D}^{-1/2}(x/x_0)a(x_0) \tag{A12}$$

It incorporates the variation caused by the spreading of the rays (the usual intensity conservation law) and an exponential decay (energy loss) mechanism defined by the subprincipal symbol.

Therefore, the GO solution for the general case is

$$u(x) = \left\{\exp\left[\nu\int_{x_0}^{x}\kappa\cdot dx\right]\mathscr{D}^{-1/2}(x/x_0)\right\}u(x_0) \tag{A13}$$

It defines the continuation of the initial solution along the ray connecting x_0 to x.

APPENDIX C. POINT MOTION IN A GRAVITATIONAL FIELD

To emphasize the connection between the trajectories of mechanics and the rays in optics, we consider the motion of a point particle of unit mass in a uniform gravitational field $g\hat{z}$. The rays in the LLP are obtained with $g = \alpha/2$.

The trajectory of the particle satisfies Newton's law:

$$\begin{cases} \ddot{x} = 0 \\ \ddot{z} = -g \end{cases} \tag{A14}$$

which has the solutions

$$\begin{cases} x = x_0 + \xi t \\ z = z_0 + \zeta_0 t - \tfrac{1}{2}gt^2 \end{cases} \tag{A15}$$

where the momenta

$$(\xi, \zeta) = \left(\frac{dx}{dt}, \frac{dz}{dt}\right) \tag{A16}$$

have the initial values

$$(\xi_0, \zeta_0) = (\xi(t = 0), \zeta(t = 0)) \tag{A17}$$

The energy of the particle

$$E = \tfrac{1}{2}(\xi^2 + \zeta^2) + gz \tag{A18}$$

is conserved along its trajectory.

Equivalently, the equations of motion can be obtained with Hamiltonian dynamics. The Hamiltonian of a particle in the gravitation field $g\hat{z}$ is

$$H = \tfrac{1}{2}(\xi^2 + \zeta^2) + gz \tag{A19}$$

The particle trajectories satisfy Hamilton's equations:

$$\partial_t(x, z, \xi, \zeta) = (\partial_\xi H, \partial_\zeta H, -\partial_x H, -\partial_z H) = (\xi, \zeta, 0, -g) \tag{A20}$$

Since $H = E$, the Hamiltonian is conserved along the trajectories.

APPENDIX D. AMPLITUDE HALF-DENSITY TRANSPORT IN A GENERAL CONTINUATION PROBLEM

Combining the amplitude transport equations (A8) and the half-density relation (107) (where the velocity field $v = p_\kappa(x, \Phi_x)$), one obtains the half-density transport equation in X for a general problem:

$$\mathfrak{L}_v \beta_X + c_0(x, \Phi_x)\beta_X = 0 \tag{A21}$$

Since v is the projection on X of the Hamilton vector field $\mathscr{H}$, the corresponding equation on the LSM Λ is the pullback (lift) of (A21) through π_X^*:

$$\mathfrak{L}_{\mathscr{H}} \beta_\Lambda + c_0(x, \kappa)\beta_\Lambda = 0 \tag{A22}$$

where the Lie derivative property

$$\pi_X^*(\mathfrak{L}_v \beta_X) = \mathfrak{L}_{(d\pi_X{}^* v)}(\pi_X^* \beta_X) = \mathfrak{L}_{\mathscr{H}} \beta_\Lambda \tag{A23}$$

has been invoked. With the half-density relation (111), the half-density transport equation on Λ for a general continuation problem is

$$\frac{d}{d\tau} h + c_0(x, \kappa)h = 0 \tag{A24}$$

It has the solution

$$h(H^\tau \lambda_0) = h(\lambda_0) \exp\left[-\int_0^\tau c_0(x, \kappa)\, d\tau\right] \tag{A25}$$

along the phase space trajectory $H^\tau \lambda_0$. The LLP and the HMP, as mentioned in Appendix B, are cases in which $c_0(x, \kappa) = 0$. Equation (A24) yields (113) immediately for these cases.

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