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The scattering of an H-polarized plane wave from an axially slotted infinite cylinder: A dual series approach

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The azimuthal current induced by an H-polarized plane wave with an arbitrary angle of incidence on an infinite cylinder with an infinite axial slot is considered. A system of dual series equations is derived from the modal expansions of the tangential field components by enforcing the electromagnetic boundary conditions. This dual series system is then solved for the modal coefficients with techniques borrowed from the Riemann-Hilbert problem of complex variable theory. The resulting infinite system of linear equations for the modal coefficients can be handled in an efficient manner. A comparison of the generalized dual series solution with a purely numerical method of moments solution based upon vector and scalar potentials is made. In contrast to this method of moments solution, it explicitly contains the behavior of the solution near the aperture rim and can generate the current values in a shadow region for small to large ratios of cylinder radius to wavelength without additional special considerations.

1. INTRODUCTION

The electromagnetic coupling problem as it applies to an enclosed region, an external source and a coupling aperture is one of major importance, both theoretically and from a practical point of view. Solutions of an analytic nature would provide insight into the coupling mechanism by which electromagnetic energy penetrates apertures into enclosed regions. Moreover, accurate solutions of problems of this type would provide standards for the evaluation of scattering codes, especially near the edge at the aperture rim where purely numerical techniques may encounter difficulties.

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Recent developments in the theory and applications of dual series equations [Casey, 1981, 1983a, b] and their relationship to the Riemann-Hilbert problem [Ziolkowski, 1983] make it possible to obtain analytic solutions to families of canonical problems descriptive of electromagnetic (and, indeed, acoustic) coupling via apertures into enclosed regions. The generalized dual series equations approach will be used in this paper to calculate the azimuthal current induced on an axially slotted cylinder by an H-polarized plane wave at an arbitrary angle of incidence. This problem has been studied by Morris [1982] who solved integral equations with highly singular

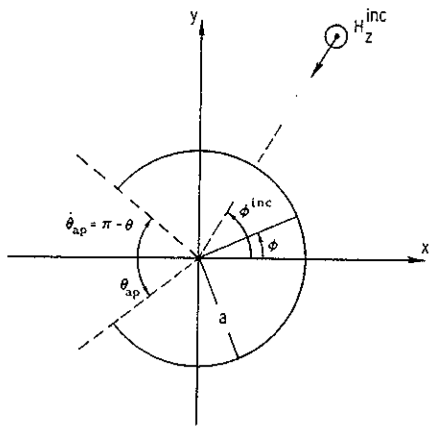

Fig. 1. Configuration of the scattering of an H-polarized plane wave from a cylinder with an infinite axial slot.

kernels using the Hadamard finite part interpretation of the integrals to obtain this current. Another alternate approach is a two-dimensional version of the method of moments solution based on vector and scalar potentials analogous to that developed by Glisson and Wilton [1980]. In contrast to Morris's method, the generalized dual series equations approach is straightforward. In contrast to the method of moments, it explicitly contains the behavior of the solution near the aperture rim.

It has recently been brought to our attention that work of a similar nature has been reported in the Russian literature. Specifically, Koshparënok and Shestopalov [1971] have applied the Riemann-Hilbert analysis of the dual series equations developed by Agronovich et al. [1962] to the axially slotted cylinder problem. In contrast to that work, this paper develops a novel approach to the solution of the infinite system of equations obtained with the Riemann-Hilbert technique. We have developed a truncation procedure that generates a general solution to that system which is not restricted to any special parameter regime. Furthermore, the current is calculated here, a quantity not considered by Koshparënok and Shestopalov [1971]. In addition, using an asymptotic form of the solution coefficients, we are able to demonstrate analytically that our general results exhibit the correct behavior near an edge of the aperture.

The structure of this paper is as follows. In section 2, the formulation of the axially slotted cylinder problem is given. The generalized dual series equations approach is then used in section 3 to obtain the solution of that problem. The currents on the cylinder are calculated in section 4. A comparison is made in section 5 between the results generated with the generalized dual series equations approach and a two-dimensional method of moments code. Conclusions and a brief summary are then given in section 6.

2. FORMULATION

The electromagnetic coupling of a plane wave through an infinite axial slot in a thin perfectly conducting infinite cylinder is considered. The magnetic field vector of the plane wave is taken to be parallel to the axis of the cylinder. This H-polarized plane wave is assumed normally incident on the cylinder; hence the problem is two dimensional. The currents it induces on the cylinder are desired.

As shown in Figure 1, a cylindrical coordinate system (ρ, ϕ, z) is centered on the axis of the cylinder; the z axis coincides with the cylinder's axis. The angle of incidence, ϕ^{inc}, of the plane wave is arbitrary. The radius of the cylinder is "a." The metallic portion of the cylinder lies in the region $-\theta < \phi < \theta$; the aperture is the complementary region $\theta < \phi < 2\pi - \theta$.

A dual series equation approach to this problem is employed. The incident and scattered fields are first decomposed into Fourier modes. The expansion coefficients are then obtained by matching the fields on the cylinder and in the aperture.

For the given polarization, Maxwell's equations decouple and only the E_ρ, E_ϕ and H_z components of the field are excited. The components of the field tangential to the surface of the aperture and the cylinder are of particular importance in the dual series formulation. They are related by

$$E_\phi = \frac{j}{\omega\epsilon}\frac{\partial}{\partial\rho} H_z \tag{1}$$

where, as throughout this paper, the time dependence $e^{j\omega t}$ has been assumed and suppressed.

The incident magnetic field may be expressed in terms of Fourier modes as

$$H_z^{inc} = \tilde{H}_o e^{jk\rho \cos(\phi-\phi^{inc})}$$

$$= \tilde{H}_o \sum_{n=-\infty}^{\infty} \left[j^{|n|} J_{|n|}(k\rho) e^{-jn\phi^{inc}} \right] e^{jn\phi} \tag{2}$$

From equation (1) it follows that

$$E_\phi^{inc} = jZ_o\tilde{H}_o \sum_{n=-\infty}^{\infty} e^{jn\phi} \left[j^{|n|} J'_{|n|}(k\rho)\, e^{-jn\phi^{inc}} \right] \tag{3}$$

where $J'_m(x) = (d/dx)J_m$ and $Z_o = k/\omega\epsilon$ is the free-space characteristic impedance. The corresponding Fourier expansions of the scattered fields are

$$H^s_{z>} = \tilde{H}_o \sum_{n=-\infty}^{\infty} e^{jn\phi}\, b_n J'_{|n|}(ka)\, H_{|n|}(k\rho) \qquad \rho > a \tag{4a}$$

$$H^s_{z<} = \tilde{H}_o \sum_{n=-\infty}^{\infty} e^{jn\phi}\, b_n H'_{|n|}(ka)\, J_{|n|}(k\rho) \qquad \rho < a \tag{4b}$$

$$E^s_{\phi>} = jZ_o\tilde{H}_o \sum_{n=-\infty}^{\infty} e^{jn\phi}\, b_n J'_{|n|}(ka)\, H'_{|n|}(k\rho) \qquad \rho > a \tag{4c}$$

$$E^s_{\phi<} = jZ_o\tilde{H}_o \sum_{n=-\infty}^{\infty} e^{jn\phi}\, b_n H'_{|n|}(ka)\, J'_{|n|}(k\rho) \qquad \rho < a \tag{4d}$$

where H_n is the Hankel function of second kind and order n and $H'_n(x) = (d/dx)H_n$. Note that (4c) and (4d) realize the continuity of the tangential component of the electric field across the interface $\rho = a$. Boundary conditions for the tangential electric and magnetic field at the surface $\rho = a$ are now enforced to obtain the dual series equations.

The total tangential electric field is zero on the metal. Equivalently, this may be expressed as the equality of the scattered and the negative of the incident electric fields there:

$$E^s_\phi(a,\phi) = -E^{inc}_\phi(a,\phi) \equiv F(\phi) \qquad |\phi| < \theta \tag{5}$$

Substituting (4c) or (4d) into (5), one obtains

$$j\tilde{H}_o Z_o \sum_{n=-\infty}^{\infty} b_n J'_{|n|}(ka)\, H'_{|n|}(ka)\, e^{jn\phi} = F(\phi) \qquad |\phi| < \theta \tag{6}$$

The dc components of the fields can be extracted from this relation by introducing the functions $Q_n(x)$ so that

$$-j\pi\, J'_o(x)H'_o(x) = 1 - Q_o(x) \tag{7a}$$

$$j\pi\, x^2 J'_n(x)H'_n(x) = n[1-Q_n(x)] \qquad n > 0 \tag{7b}$$

where $Q_n(0) \equiv 0$. Thus (6) becomes

$$\sum_{n=-\infty}^{\infty} |n|\, [1-Q_{|n|}(ka)]\, e^{jn\phi}\, b_n = (ka)^2 [1-Q_o(ka)]\, b_o + \frac{(ka)^2\pi}{Z_o} F(\phi) \qquad |\phi| < \theta \tag{8}$$

On the other hand, continuity of H_ϕ across the aperture and the Wronskian relationship

$$J'_{|n|}(ka)H_{|n|}(ka) - J_{|n|}(ka)H'_{|n|}(ka) = \frac{2j}{\pi ka} \quad (9)$$

give

$$\sum_{n=-\infty}^{\infty} b_n e^{jn\phi} = 0 \qquad \theta < |\phi| \quad (10)$$

Equations (8) and (10) constitute a system of dual series equations that can be solved to obtain the unknown modal amplitudes b_n.

3. SOLUTION OF THE DUAL SERIES EQUATIONS

The solution of the dual series equations (8) and (10) may be obtained by first solving an associated static problem. That static problem is defined by

$$\sum_{n=-\infty}^{\infty} b_n e^{jn\phi} = 0 \qquad \theta < |\phi| \quad (11a)$$

and

$$\sum_{n=-\infty}^{\infty} b_n |n| e^{jn\phi} = \alpha b_o + f(e^{j\phi})$$

$$|\phi| < \theta \quad (11b)$$

where the forcing function f is assumed to have the Fourier expansion

$$f(e^{j\phi}) = \sum_{n=-\infty}^{\infty} f_n e^{jn\phi} \quad (12)$$

Solution of this problem using Riemann-Hilbert problem techniques follows directly from Agranovich et al. [1962] or Ziolkowski [1983]. The analogous problem of the scattering of a plane wave by an infinite diffraction grating was considered by Agranovich et al. [1962]. As discussed by Ziolkowski [1983], the Riemann-Hilbert approach explicitly accounts for the singularity of the fields near the edge of the aperture. Assuming that

$$x_m = m\, b_m \quad (13)$$

the result is

$$x_m = \alpha V_m^o b_o + \sum_{n=-\infty}^{\infty} f_n V_m^n + 2cR_m$$

$$m \neq 0 \quad (14a)$$

$$0 = \alpha V_o^o b_o + \sum_{n=-\infty}^{\infty} f_n V_o^n + 2cR_o \quad (14b)$$

$$-b_o = \alpha W^o b_o + \sum_{n=-\infty}^{\infty} f_n W^n + 2cS \quad (14c)$$

The terms V_m^n, W^m, R_m, and S result from combinations of Legendre functions and are given in Appendix A. The auxiliary constant "c," which is introduced in the solution procedure, is associated with the behavior of the solution at infinity. It can be shown that $c = x_{-1}$. The evaluation of the expression V_m^m requires special consideration; it is carried out in Appendix B.

The solution to the time harmonic problem, equations (8) and (10), may be obtained by making the identifications:

$$f(e^{j\phi}) = \sum_{n=-\infty}^{\infty} |n|\, Q_{|n|}(ka)\, b_n e^{jn\phi}$$

$$+ \frac{(ka)^2 \pi}{Z_o} F(\phi) \quad (15)$$

and

$$\alpha = (ka)^2 [1 - Q_o(ka)] \quad (16)$$

Equations (14a)-(14c) then yield the

infinite linear system

$$x_m = (ka)^2 \, [1-Q_o(ka)] \, V_m^o \, b_o + \sum_{n=-\infty}^{\infty} \frac{|n|}{n} Q_{|n|}(ka) \, V_m^n \, x_n + \frac{(ka)^2 \pi}{Z_o} \sum_{n=-\infty}^{\infty} F_n \, V_m^n + 2cR_m \qquad m \neq 0 \tag{17a}$$

$$0 = (ka)^2 \, [1-Q_o(ka)] \, V_o^o \, b_o + \sum_{n=-\infty}^{\infty} \frac{|n|}{n} Q_{|n|}(ka) \, V_o^n \, x_n + \frac{(ka)^2 \pi}{Z_o} \sum_{n=-\infty}^{\infty} F_n \, V_o^n + 2cR_o \tag{17b}$$

$$-b_o = (ka)^2 \, [1-Q_o(ka)] \, W^o \, b_o + \sum_{n=-\infty}^{\infty} \frac{|n|}{n} Q_{|n|}(ka) \, W^n \, x_n + \frac{(ka)^2 \pi}{Z_o} \sum_{n=-\infty}^{\infty} F_n \, W^n + 2cS \tag{17c}$$

where from (3) and (5) the Fourier coefficients of F are

$$F_n = -j^{|n|+1} \, Z_o \, \tilde{H}_o \, J'_{|n|}(ka) \, e^{-jn\phi^{inc}} \tag{18}$$

The infinite system of equations (17a)-(17c) may be treated in several ways. It has been found that truncating F_n and $Q_{|n|}$ for $|n|$ greater than some value N and using Gauss elimination to solve the remaining finite system yields good numerical approximations for the coefficients c, b_o, $x_{\pm 1}$, •••, $x_{\pm N}$. The remaining coefficients, x_m, $|m| > N$, are given by the expression

$$x_m = (ka)^2 \, [1-Q_o(ka)] \, V_m^o \, b_o + \sum_{n=-N}^{N} \frac{|n|}{n} Q_{|n|}(ka) \, V_m^n \, x_n + \frac{(ka)^2 \pi}{Z_o} \sum_{n=-N}^{N} F_n \, V_m^n + 2cR_m \tag{19}$$

As N approaches infinity, this approximation scheme becomes exact.

4. EVALUATION OF THE CURRENT INDUCED ON THE SURFACE OF THE SLITTED CYLINDER

Although the current induced on the cylinder's surface may be readily computed with the coefficients defined by (17) and (19), an alternate expression, particularly useful in studying the behavior of the currents near an edge of the aperture, speeds up the summation process. The current induced on the cylinder may be expressed with (4a), (4b) and (9) as

$$J_\phi(a,\phi) = H_{z<}(\phi) - H_{z>}(\phi)$$

$$= \frac{2\tilde{H}_o}{j\pi ka} \left[b_o + \sum_{m\neq 0} \frac{x_m}{m} e^{jm\phi} \right] \tag{20}$$

where $\Sigma_{m\neq 0}$ means to sum from $m = -\infty$ to $m = +\infty$ except the term with $m = 0$. The rate of convergence of this infinite sum can be increased by expressing it as

$$\sum_{m\neq 0} \frac{x_m}{m} e^{jm\phi} = \sum_{m\neq 0} \frac{x_m - \tilde{x}_m}{m} e^{jm\phi} + S_e \tag{21}$$

where the term $\tilde{x}_m$ is a large m approximation of x_m and

$$S_e = \sum_{m\neq 0} \frac{\tilde{x}_m}{m} e^{jm\phi} \tag{22}$$

The first sum on the right-hand side of (21) is rapidly converging; the second sum, S_e, is obtained analytically as follows:

From Appendix A, the equation

$$R_m = \frac{1}{2} P_m(\cos\theta) \tag{23}$$

and the large $|m|$ approximation of V_m^n :

$$V_m^n \approx \frac{1}{2}\Big[P_m(\cos\theta)P_{n+1}(\cos\theta) - P_{m+1}(\cos\theta)P_n(\cos\theta)\Big] \qquad |n| \le N \tag{24}$$

are obtained. The relations allow one to write the term $\tilde{x}_m$, as defined by (19), in terms of the Legendre functions P_m and P_{m+1} as

$$\tilde{x}_m = \kappa_1 P_m(\cos\theta) + \kappa_2 P_{m+1}(\cos\theta) \tag{25}$$

where the coefficients

$$\kappa_1 = \frac{1}{2}\Bigg[(ka)^2(1-Q_o)\, b_o \cos\theta + 2c + \frac{(ka)^2\pi}{Z_o}\sum_{n=-N}^{N} F_n\, P_{n+1}(\cos\theta) + \sum_{\substack{n=-N\\ n\neq 0}}^{N} \frac{|n|}{n} Q_{|n|}\, x_n\, P_{n+1}(\cos\theta)\Bigg] \tag{26}$$

and

$$\kappa_2 = -\frac{1}{2}\Bigg[(ka)^2(1-Q_o)b_o + \frac{(ka)^2\pi}{Z_o}\sum_{n=-N}^{N} F_n\, P_n(\cos\theta) + \sum_{\substack{n=-N\\ n\neq 0}}^{N} \frac{|n|}{n} Q_{|n|}\, x_n\, P_n(\cos\theta)\Bigg] \tag{27}$$

are independent of m. As shown in Appendix C, the sum S_e then reduces to the analytical expression:

$$\begin{aligned} S_e &= \sum_{m\neq 0} \frac{\tilde{x}_m}{m} e^{jm\phi} = \kappa_1 \sum_{m\neq 0} \frac{P_m(\cos\theta)e^{jm\phi}}{m} + \kappa_2 \sum_{m\neq 0} \frac{P_{m+1}(\cos\theta)e^{jm\phi}}{m} \\ &= \kappa_1\Bigg\{-2\ln\left|\cos\frac{\phi}{2} + \left(\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}\right)^{1/2}\right| + 2i\left[\sin^{-1}\left(\frac{\sin\frac{\phi}{2}}{\sin\frac{\theta}{2}}\right) - \frac{\phi}{2}\right]\Bigg\} + \kappa_2\Bigg\{2\sin^2\frac{\theta}{2} \\ &\quad + 4\cos^2\frac{\theta}{2}\ln\left|\cos\frac{\theta}{2}\right| - 2\cos\theta\ln\left|\cos\frac{\phi}{2} + \left(\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}\right)^{1/2}\right| - 4\cos\frac{\phi}{2}\left(\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}\right) - \sum_{\substack{m\neq 0\\ m\neq 1}} (-)^m \frac{P_m(\cos\theta)}{m(m+1)} \\ &\quad + i\left[2\cos\theta\left[\sin^{-1}\left(\frac{\sin\frac{\phi}{2}}{\sin\frac{\theta}{2}}\right) - \frac{\phi}{2}\right] + 4\sin\frac{\phi}{2}\left(\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}\right)^{1/2}\right]\Bigg\} \end{aligned} \tag{28}$$

The only nonclosed expression, the sum in the κ_2 coefficient, has terms decreasing in magnitude as $m^{-5/2}$ for large m; hence it is rapidly convergent and easy to evaluate. Note that the sum S_e depends on the term

$$\left(\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}\right)^{1/2} = (\cos\phi - \cos\theta)^{1/2}$$

which is characteristic of the behavior of the current near an edge. This sum dominates the current variation near an edge; hence the first sum in (21) is

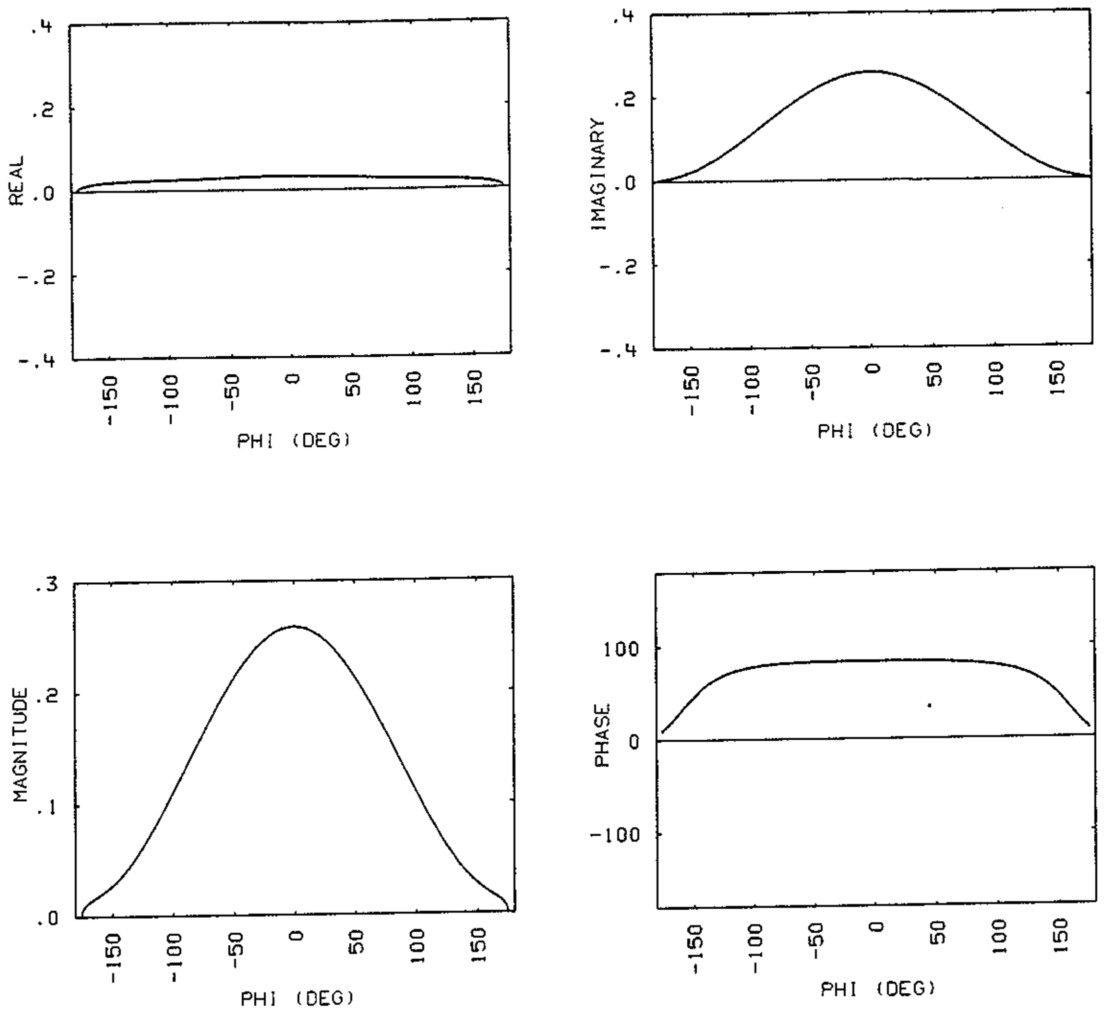

Fig. 2. Currents calculated by the dual series (solid curve) and the method of moments (dotted curve) for an H-polarized plane wave incident at $\phi^{inc} = 180°$ on a cylinder of radius 0.01λ with an aperture angle $\theta_{ap} = 5°$.

numerically well behaved there, and the total current is readily calculated. Similarly, the fields can be shown to satisfy the correct edge behavior.

5. SAMPLE NUMERICAL RESULTS

Currents generated by the dual series technique may be used to assess the accuracy of the method of moments solution scheme described in Appendix D, especially since the dual series solution explicitly contains the correct current behavior near the edges. The dual series system (17) is truncated by neglecting $Q_{|n|}$ and F_n for $|n| >$ N. Also, the first sum on the right of (21) is truncated for $|m| > M$. Both truncation indices have been chosen sufficiently large in all of the following cases so that the dual series results have converged to the exact solution. Current values are obtained from the moment method solution at a discrete set of points. As illustrated in Figure 7, these points correspond to the aperture endpoints and to the nodes of the triangular expansion functions

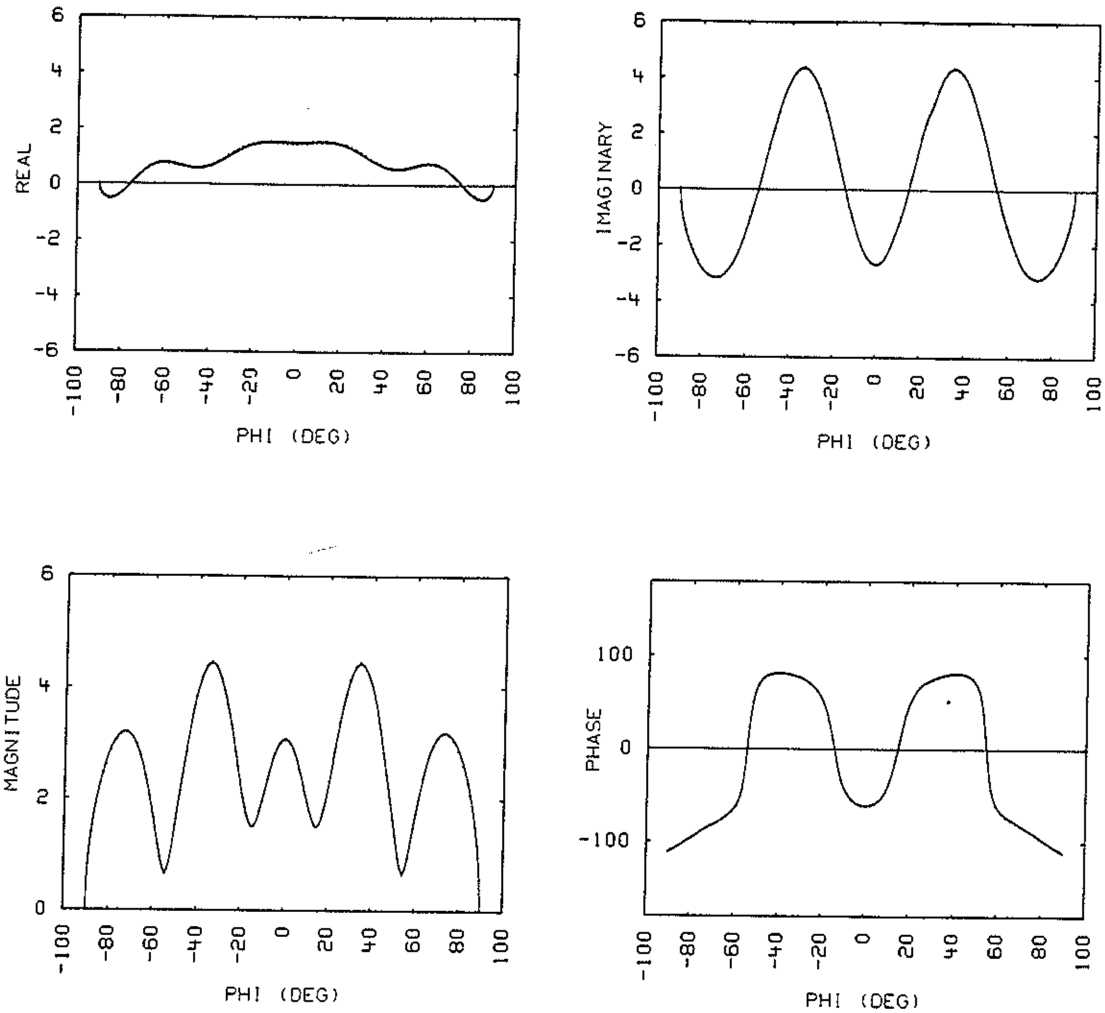

Fig. 3. Currents calculated by the dual series (solid curve) and the method of moments (dotted curve) for an H-polarized plane wave incident at ϕ^{inc} = 180° on a cylinder of radius 1λ with an aperture angle θ_{ap} = 90°.

employed in this solution. To facilitate adequate tracking of the currents near an edge, a nonuniform spacing of these points is chosen. In particular, the distances between the five points nearest each edge are chosen uniformly as d_1; the distances between the next five points are uniformly d_2; and the distances between the remaining points are uniformly d_3. The ratios of d_1 and d_2 to d_3, the spacing ratios, are denoted by s_1 and s_2 respectively. The number of unknowns employed in the moment method solution is denoted as N_{mom}. In all cases the incident magnetic field is normalized to unity at the center of the cylinder.

Currents on an electrically thin (quasi-static) cylinder with a total aperture angle of 10° are illustrated in Figure 2. Currents computed from the dual series technique are denoted by solid curves, those from the method of moments by dotted curves. The geometric parameters (Figure 1) are θ_{ap} = 5°, ϕ^{inc} = 180°, and a = 0.01λ. The

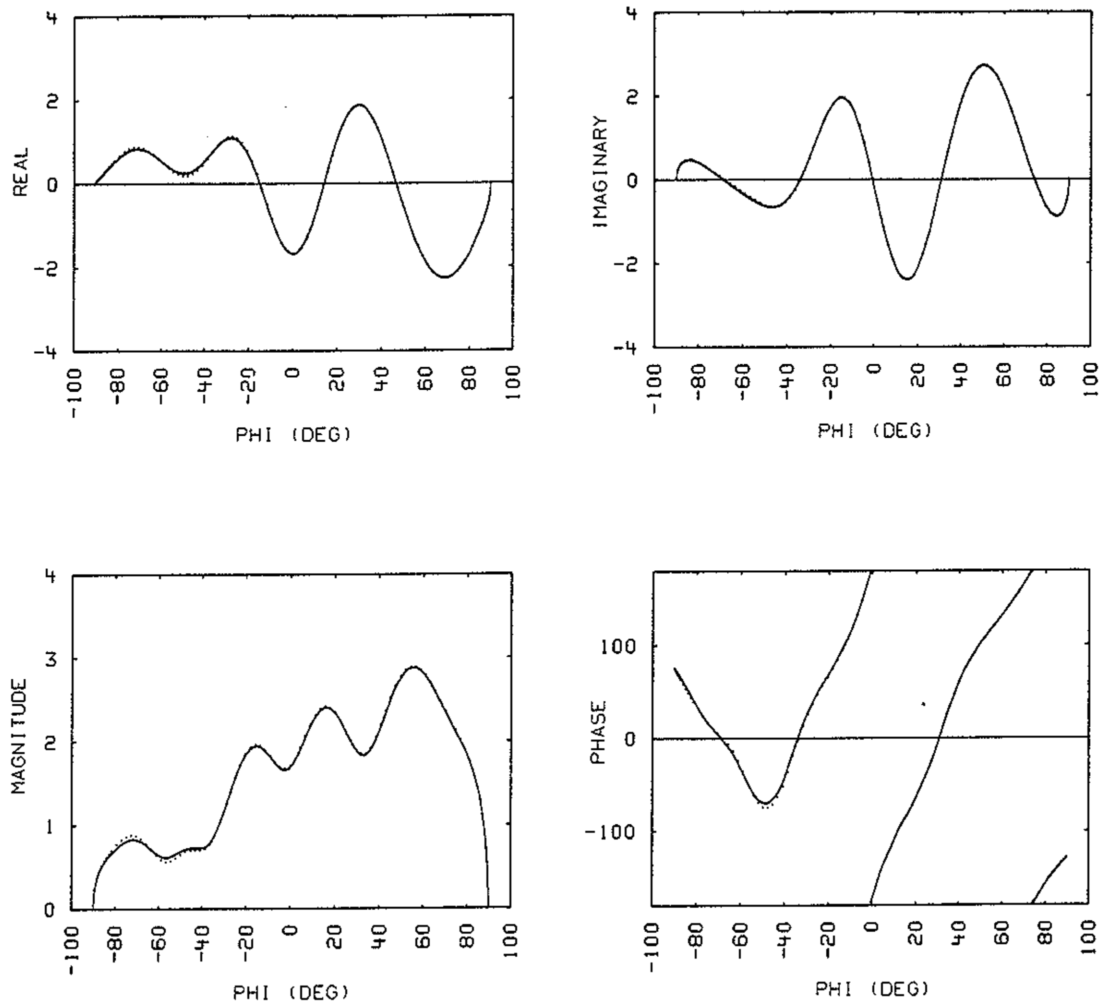

Fig. 4. Currents calculated by the dual series (solid curve) and the method of moments (dotted curve) for an H-polarized plane wave incident at ϕ^{inc} = 90° on a cylinder of radius 1λ with an aperture angle θ_{ap} = 90°.

truncation indices in the dual series solution were set at N = 10 and M = 20. The number of unknowns in the moment method solution, N_{mom}, was 45, with spacing ratios s_1 = 0.25 and s_2 = 0.5. For this case, the moment method is indistinguishable from the dual series results.

In Figures 3 and 4 the angles of incidence of the plane wave are, respectively, ϕ^{inc} = 180° and ϕ^{inc} = 90°. Note that the respective incoming fields strike the aperture center and the aperture edge normally. The remaining parameters common to both figures are θ_{ap} = 90°, a = 1λ, N = 25, M = 250, N_{mom} = 91, s_1 = 0.25, and s_2 = 0.5. The physical optics nature of the current begins to appear in the lit region of the magnitude plot in Figure 4. Note that the method of moments solution begins to have difficulty tracking the exact (dual series) solution in the shadow regions of Figure 4.

Similar results are shown in Figures 5

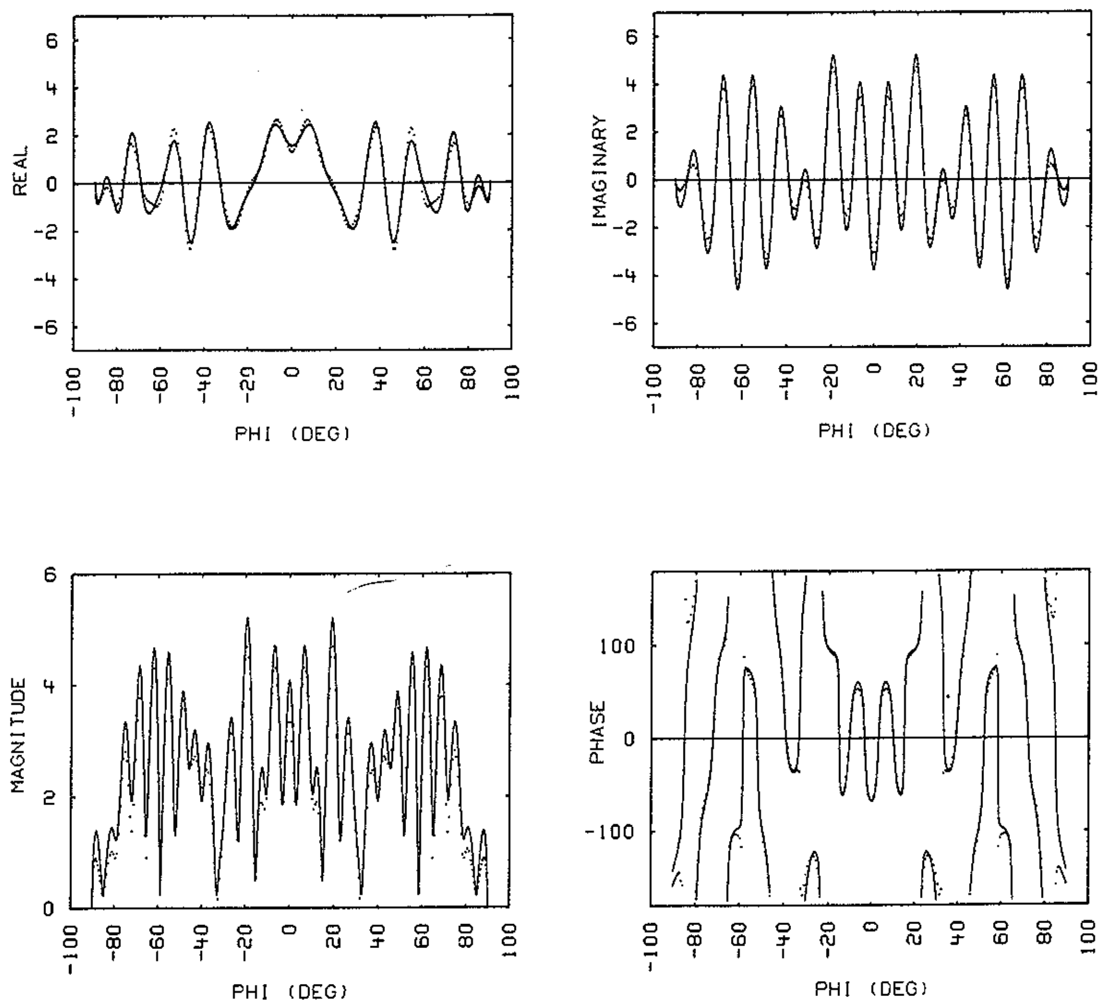

Fig. 5. Currents calculated by the dual series (solid curve) and the method of moments (dotted curve) for an H-polarized plane wave incident at ϕ^{inc} = 180° on a cylinder of radius 5λ with an aperture angle θ_{ap} = 90°.

and 6 for a cylinder of radius 5λ. The appropriate parameters are: θ_{ap} = 90°, N = 65, M = 400, s_1 = 0.5, and s_2 = 0.8. The number of unknowns in the moment method solution is 251 and 301, respectively, for Figures 5 and 6. Finer tracking is required in these moment method solutions, especially near the edge in the shadow region, before accurate results will be obtained. Again, the physical optics characteristics of the current are apparent in the magnitude plot of Figure 6.

6. CONCLUSIONS AND REMARKS

A dual series solution to the problem of the scattering of an H-polarized plane wave from a slitted cylinder serves as an excellent test case for two-dimensional scattering codes. The method of moments code developed in Appendix D contains all of the essential features of a two-dimensional analogue of the surface patch code developed by Rao et al. [1982]. A comparison of the results shows that accurate method of

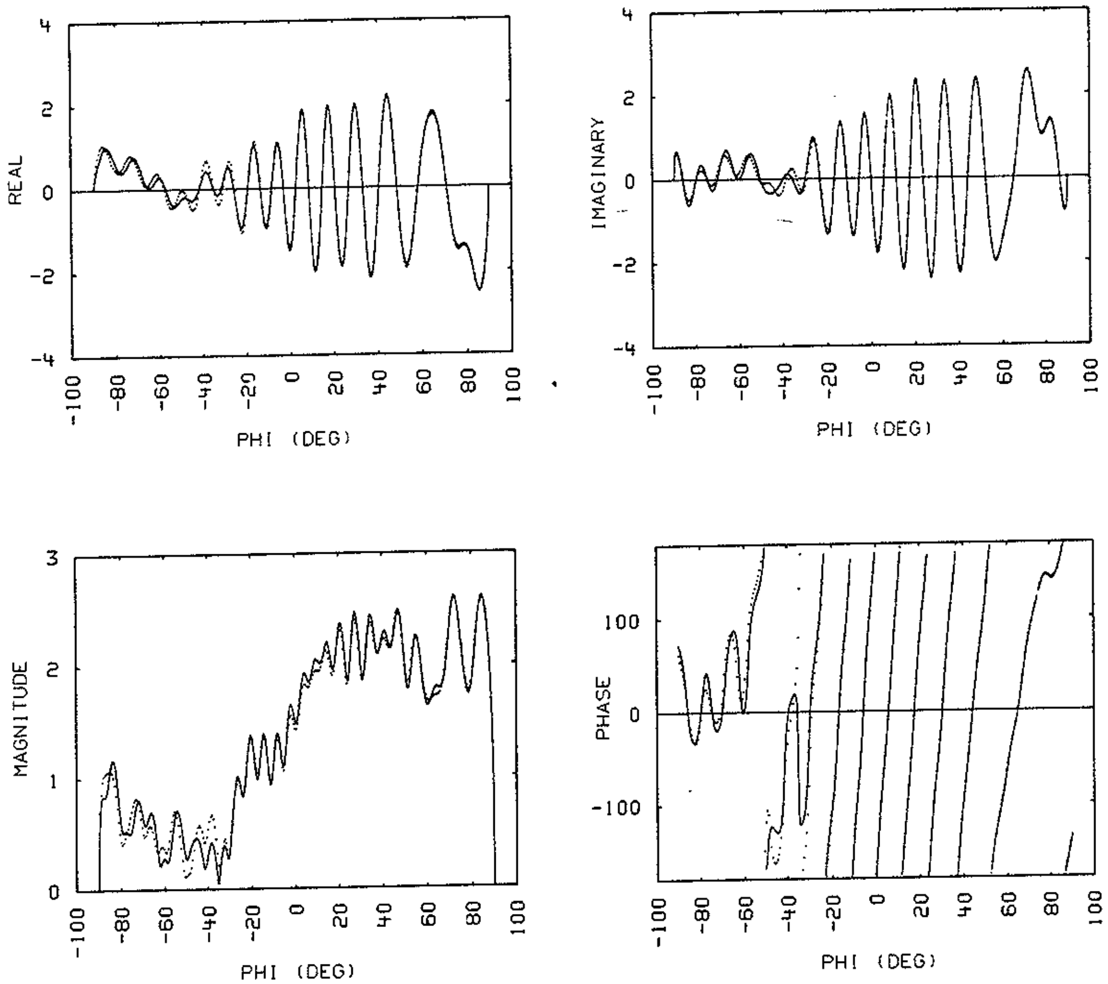

Fig. 6. Currents calculated by the dual series (solid curve) and the method of moments (dotted curve) for an H-polarized plane wave incident at $\phi^{inc} = 90°$ on a cylinder of radius 5λ with an aperture angle $\theta_{ap} = 90°$.

moment solutions require fine gridding near an edge, especially if that edge lies in a shadow region.

Dual series techniques apply to a wide variety of coupling problems in separable geometries. In particular, the scattering of an electromagnetic plane wave with an arbitrary angle of incidence from a thin spherical shell with a circular aperture is currently being studied with these techniques. This should provide an excellent test case for three-dimensional scattering codes.

APPENDIX A: COEFFICIENTS OF THE SOLUTION SYSTEM OF EQUATIONS

$$V_m^n = \frac{m+1}{2(m-n)} [P_m(u)P_{n+1}(u) - P_{m+1}(u)P_n(u)]$$

$$n \neq -1, \ m \neq n \qquad \text{(A1a)}$$

$$V_m^n = \frac{1}{2} [P_m(u) - P_{m+1}(u)] \qquad n = -1 \qquad \text{(A1b)}$$

$$V_m^n \qquad m = n \neq -1 \qquad \text{(A1c)}$$

See Appendix B for discussion of (A1c).

$$R_m = \frac{1}{2} P_m(u) \tag{A2}$$

$$W^n = -\frac{1}{2} \mu_{n+1}(u) \ln\left(\frac{1+u}{2}\right) + \frac{1}{2n} [P_n(u) - P_{n-1}(u)] \qquad n \geq 1 \tag{A3a}$$

$$W^n = \left(\frac{1+u}{2}\right) \ln\left(\frac{1+u}{2}\right) \qquad n = 0 \tag{A3b}$$

$$W^n = \left(\frac{u-1}{2}\right)\left[1 + \ln\left(\frac{1+u}{2}\right)\right] \qquad n = -1 \tag{A3c}$$

$$W^n = -\frac{1}{2} \mu_{-n}(u) \ln\left(\frac{1+u}{2}\right) - \frac{1}{2n} [P_{-n}(u) - P_{-n-1}(u)] \qquad n < -1 \tag{A3d}$$

$$S = \frac{1}{2} \sum_{m \neq 0} (-)^m \frac{P_m(u)}{m} = -\frac{1}{2} \ln\left(\frac{1+u}{2}\right) \tag{A4}$$

where

$$u = \cos\theta$$

$$\mu_o(u) = 1$$

$$\mu_1(u) = -u$$

$$\mu_n(u) = P_n(u) - 2uP_{n-1}(u) + P_{n-2}(u) \qquad n \geq 2$$

APPENDIX B: EVALUATION OF V_n^n

The values of the expression V_m^n when m = n are desired. The cases $n \geq 0$, $n = -1$ and $n \leq -2$ must be considered separately. Only the second is simple; the others depend on an application of L'Hospital's rule. From Appendix A, the application of L'Hospital's rule to (A1) yields

$$V_n^n = \frac{n+1}{2} \left[P_{n+1}(\cos\theta) \frac{\partial}{\partial v} P_v(\cos\theta)\Big|_{v=n} - P_n(\cos\theta) \frac{\partial}{\partial v} P_v(\cos\theta)\Big|_{v=n+1} \right] \tag{B1}$$

The derivative of a Legendre polynomial with respect to its index evaluated at a nonnegative index may be shown by induction to be

$$\frac{\partial}{\partial v} P_v(\cos\theta)\Big|_{v=m} = 2P_m(\cos\theta)\ln\left(\cos\frac{\theta}{2}\right) + U_m(\cos\theta) \tag{B2}$$

where

$$U_o(u) = 0 \tag{B3a}$$

$$U_1(u) = u - 1 \tag{B3b}$$

and

$$U_{m+1}(u) = \frac{uP_m(u) - P_{m-1}(u)}{(m+1)^2} + \frac{2m+1}{m+1} uU_m(u) - \frac{m}{m+1} U_{m-1}(u) \tag{B3c}$$

Thus, for $n \geq 0$ the desired expression is

$$V_n^n = \frac{n+1}{2} \left[P_{n+1}(\cos\theta)\, U_n(\cos\theta) - P_n(\cos\theta)\, U_{n+1}(\cos\theta) \right] \tag{B4}$$

With the relation

$$P_{-s}(u) = P_{s-1}(u) \tag{B5}$$

it follows simply from (A1) that for $n = -1$,

$$V_{-1}^{-1} = 0 \tag{B6}$$

Finally, for $m, n < -1$, one may use

(B5) to express V_n^m as

$$V_m^n = \frac{m+1}{2(m-n)}\Big[P_{-m-1}(\cos\theta)P_{-n-2}(\cos\theta) - P_{-m-2}(\cos\theta)P_{-n-1}(\cos\theta)\Big] \qquad \text{(B7)}$$

Application of L'Hospital's rule to (B7) yields, for $n \leq -2$,

$$V_n^n = \frac{n+1}{2}\Big[P_{-n-1}(\cos\theta)\,\frac{\partial}{\partial v}P_v(\cos\theta)\Big|_{v=-n-2} - P_{-n-2}(\cos\theta)\,\frac{\partial}{\partial v}P_v(\cos\theta)\Big|_{v=-n-1}\Big]$$

$$= \frac{n+1}{2}\Big[P_{-n-1}(\cos\theta)U_{-n-2}(\cos\theta) - P_{-n-2}(\cos\theta)U_{-n-1}(\cos\theta)\Big] \qquad \text{(B8)}$$

APPENDIX C: EVALUATION OF THE SUM S_e

The sum

$$S_e = \sum_{m\neq 0} \frac{\tilde{x}_m}{m} e^{jm\phi} = \kappa_1 S_1 + \kappa_2 S_2 \qquad \text{(C1)}$$

where

$$S_1 = \sum_{m\neq 0} \frac{P_m(\cos\theta)}{m} e^{jm\phi} \qquad \text{(C2)}$$

and

$$S_2 = \sum_{m\neq 0} \frac{P_{m+1}(\cos\theta)}{m} e^{jm\phi} \qquad \text{(C3)}$$

will be evaluated in this appendix. This evaluation makes use of the following four integrals:

$$\int_{-\theta}^{\phi} \frac{\cos\frac{\psi}{2}\,d\psi}{(\cos\psi-\cos\theta)^{1/2}} = 2^{1/2}\sin^{-1}\left(\frac{\sin\frac{\phi}{2}}{\sin\frac{\theta}{2}}\right) + \frac{\pi}{2^{1/2}} \qquad \text{(C4)}$$

$$\int_{-\theta}^{\phi} \frac{\sin\frac{\psi}{2}\,d\psi}{(\cos\psi-\cos\theta)^{1/2}} = 2^{1/2}\ln\left|\cos\frac{\theta}{2}\right| - 2^{1/2}\ln\left|\cos\frac{\phi}{2} + \left(\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}\right)^{1/2}\right| \qquad \text{(C5)}$$

$$\int_{-\theta}^{\phi} \frac{\left(1-4\sin^2\frac{\psi}{2}\right)\cos\psi\,d\psi}{(\cos\psi-\cos\theta)^{1/2}} = 2^{1/2}\left\{2\sin\frac{\phi}{2}\left(\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}\right)^{1/2} + \cos\theta\left[\sin^{-1}\left(\frac{\sin\frac{\phi}{2}}{\sin\frac{\theta}{2}}\right) + \frac{\pi}{2}\right]\right\} \qquad \text{(C6)}$$

$$\int_{-\theta}^{\phi} \frac{\left(4\cos^2\frac{\psi}{2} - 1\right)\sin\frac{\psi}{2}\,d\psi}{(\cos\psi-\cos\theta)^{1/2}} = -2^{1/2}\cos\theta\,\ln\left|\frac{\cos\frac{\phi}{2} + \left(\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}\right)^{1/2}}{\cos\frac{\theta}{2}}\right| - 2^{3/2}\cos\frac{\phi}{2}\left(\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}\right)^{1/2} \qquad \text{(C7)}$$

and the identity [Oberhettinger, 1973]

$$\sum_{m=-\infty}^{\infty} P_m(\cos\theta)e^{jm\phi} = 2^{1/2}e^{-j\frac{\phi}{2}}\begin{cases}(\cos\phi-\cos\theta)^{-1/2} & -\theta < \phi < \theta < \pi \\ 0 & \theta < \phi < 2\pi - \theta\end{cases} \qquad \text{(C8)}$$

The first sum, S_1, may be expressed as

$$S_1 = i\int_{-\pi}^{\phi} \sum_{m\neq 0} P_m(\cos\theta)\, e^{im\psi}\, d\psi + \sum_{m\neq 0} (-)^m \frac{P_m(\cos\theta)}{m} \qquad \text{(C9)}$$

With Equations (A4) and (C8) one obtains

$$S_1 = -\ln\left(\frac{1+\cos\theta}{2}\right) - i(\phi+\pi) + i\, 2^{1/2}\int_{-\theta}^{\phi} \frac{e^{-i\psi/2}\, d\psi}{(\cos\psi-\cos\theta)^{1/2}} \qquad \text{(C10)}$$

Substitution of (C4) and (C5) into (C10) yields

$$S_1 = -2\ln\left|\cos\frac{\phi}{2} + \left(\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}\right)^{1/2}\right| + 2i\left[\sin^{-1}\left(\frac{\sin\frac{\phi}{2}}{\sin\frac{\theta}{2}}\right) - \frac{\phi}{2}\right] \qquad \text{(C11)}$$

The second sum, S_2, can also be represented as

$$S_2 = i\int_{-\pi}^{\phi} \sum_{m\neq 0} P_{m+1}(\cos\theta)\, e^{jm\psi}\, d\psi + \sum_{m\neq 0} \frac{(-)^m P_{m+1}(\cos\theta)}{m} \qquad \text{(C12)}$$

Again, with the aid of Appendix A, the last sum in (C12) may be expressed as

$$T = \sum_{m\neq 0} \frac{(-)^m P_{m+1}(\cos\theta)}{m}$$

$$= 1 - \cos\theta - \sum_{\substack{m\neq 0\\ m\neq 1}} (-)^m P_m(\cos\theta) \left[\frac{1}{m-1} - \frac{1}{m}\right] - \sum_{m\neq 0} (-)^m \frac{P_m(\cos\theta)}{m}$$

$$= 1 - \cos\theta - \sum_{\substack{m\neq 0\\ m\neq 1}} \frac{(-)^m P_m(\cos\theta)}{m(m-1)} + \ln\left(\frac{1+\cos\theta}{2}\right) \qquad \text{(C13)}$$

In proceeding to evaluate the remaining term in (C12), consider the expression

$$\sum_{m\neq 0} P_{m+1}(\cos\theta) e^{jm\psi} = e^{-j\psi} \sum_{m\neq 1} P_m(\cos\theta) e^{jm\psi}$$

$$= -\cos\theta + 2^{1/2} e^{-j3\psi/2} \begin{cases} (\cos\psi-\cos\theta)^{1/2} & -\theta < \phi < \theta < \pi \\ 0 & \theta < \phi < 2\pi - \theta \end{cases} \qquad \text{(C14)}$$

It now follows that (C12) becomes

$$S_2 = -i(\phi+\pi)\cos\theta + i2^{1/2}\int_{-\theta}^{\phi} \frac{e^{-j3\psi/2}\, d\psi}{(\cos\psi-\cos\theta)^{1/2}} + T \qquad \text{(C15)}$$

The identity

$$\exp\left(-j\,\frac{3}{2}\,\psi\right) = \cos\frac{\psi}{2}\left(1-4\sin^2\frac{\psi}{2}\right) - i\sin\frac{\psi}{2}\left(4\cos^2\frac{\psi}{2} - 1\right) \qquad \text{(C16)}$$

and (C6) and (C7) then yield the result

$$S_2 = 2\sin^2\frac{\theta}{2} + 4\cos^2\frac{\theta}{2}\ln\left|\cos\frac{\theta}{2}\right| - 2\cos\theta\ln\left|\cos\frac{\phi}{2} + \left(\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}\right)^{1/2}\right|$$

$$- 4\cos\frac{\phi}{2}\left(\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}\right)^{1/2}$$

$$- \sum_{\substack{m\neq 0\\ m\neq 1}} \frac{(-)^m P_m(\cos\theta)}{m(m-1)} + 2i\left\{2\sin\frac{\phi}{2}\right.$$

$$\left(\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}\right)^{1/2} + \cos\theta$$

$$\left.\left[\sin^{-1}\left(\frac{\sin\frac{\phi}{2}}{\sin\frac{\theta}{2}}\right) - \frac{\phi}{2}\right]\right\} \quad \text{(C17)}$$

The sum (C1), as given by (28), follows immediately.

APPENDIX D: A METHOD OF MOMENTS SOLUTION

The method of moments technique used to generate the numerical solutions for the currents in the comparisons given in section 5 is briefly described. The solution scheme is a Galerkin technique based on the use of vector and scalar potentials. It is a two-dimensional analogue of the electric field integral equation (EFIE) surface patch work of Rao et al. [1982].

Note that for the polarization and problem geometry discussed in section 2, only the E_ρ, E_ϕ, and H_z components of the electromagnetic field are excited. The electric field scattered by the cylinder may be expressed as

$$\bar{E}^s = - j\omega\bar{A} - \nabla\Phi \quad \text{(D1)}$$

where the scalar potential is given by

$$\Phi = \frac{a}{4j\varepsilon}\int_{-\theta}^{\theta} H_o^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\; q(\phi')\, d\phi' \quad \text{(D2)}$$

and the nonzero components of the vector potential are given by

$$A_\phi = \frac{\mu a}{4j}\int_{-\theta}^{\theta} H_o^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\cos(\phi-\phi')$$

$$J_{\phi'}(\phi')\, d\phi' \quad \text{(D3a)}$$

and

$$A_\rho = \frac{\mu a}{4j}\int_{-\theta}^{\theta} H_o^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\sin(\phi-\phi')$$

$$J_{\phi'}(\phi')\, d\phi' \quad \text{(D3b)}$$

In (D2), (D3a), and (D3b), the quantities q and $J_{\phi'}$ are, respectively, surface charge and current densities which reside on the cylinder.

The EFIE is obtained by setting E_ϕ^s to $-E_\phi^{inc}$ on the metallic portion of the cylinder's surface. This may be expressed as

$$j\omega A_\phi + \frac{1}{a}\frac{\partial}{\partial\phi}\Phi = E_\phi^{inc} \quad \text{(D4)}$$

on the metal. A simplification of (D2)-(D3b) occurs when both the source and the observer lie on the surface of the cylinder since

$$|\bar{\rho}-\bar{\rho}'| = 2a\sin\left|\frac{\phi-\phi'}{2}\right| \quad \text{(D5)}$$

A further simplification takes advantage of the two-dimensional nature of this problem. The continuity equation

$$j\omega q + \frac{1}{a}\frac{\partial}{\partial\phi}J_\phi = 0 \quad \text{(D6)}$$

may be used to express the scalar potential as

$$\Phi = \frac{j}{\omega\varepsilon\mu a}\frac{\partial}{\partial\phi}F \quad \text{(D7)}$$

where

$$F = \frac{\mu a}{4j}\int_{-\theta}^{\theta} H_o^{(2)}\left(2ka\sin\left|\frac{\phi-\phi'}{2}\right|\right)$$

$$J_{\phi'}(\phi')\, d\phi' \quad \text{(D8)}$$

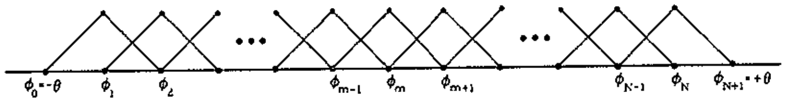

Fig. 7. Expansion and testing functions for the method of moments solution scheme.

Thus, (D4) becomes

$$\frac{j}{\omega\varepsilon\mu a^2}\left(\tilde{k}^2 A_\phi + \frac{\partial^2}{\partial\phi^2} F\right) = E_\phi^{inc} \quad \text{(D9)}$$

where

$$\tilde{k} = ka \quad \text{(D10)}$$

To obtain a numerical solution of (D9), the ϕ domain $[-\theta, \theta]$ is covered by N overlapping triangles as illustrated in Figure 7. These triangular functions are defined as

$$T_m(\phi) = \frac{\Delta_m - |\phi-\phi_m|}{\Delta_m}$$

$$\phi\varepsilon[\phi_{m-1},\phi_m] \quad \text{(D11a)}$$

$$T_m(\phi) = \frac{\Delta_{m+1} - |\phi-\phi_m|}{\Delta_{m+1}}$$

$$\phi\varepsilon[\phi_m,\phi_{m+1}] \quad \text{(D11b)}$$

$$T_m(\phi) = 0 \quad \text{otherwise} \quad \text{(D11c)}$$

with

$$\Delta_m = \phi_m - \phi_{m-1} \quad \text{(D12)}$$

Defining the inner product by

$$< f,g > = \int_{-\theta}^{\theta} f(\phi)\, g^*(\phi)\, d\phi \quad \text{(D13)}$$

the inner product of (D9) and T_m yields

$$\frac{j}{\omega\varepsilon\mu a^2}\left(\tilde{k}^2 < A_\phi, T_m > + < \frac{\partial^2}{\partial\phi^2} F, T_m >\right)$$

$$= < E_\phi^{inc}, T_m > \quad \text{(D14)}$$

Since the incident field and A_ϕ are continuous, we approximate

$$< G,T_m > \approx \frac{\Delta_m + \Delta_{m+1}}{2} G(\phi_m) \quad \text{(D15)}$$

where G equals either A_ϕ or E_ϕ^{inc}. With the application of integration by parts and this approximation, yields (D14)

$$\frac{j}{\omega\varepsilon\mu a}\left\{\frac{F(\phi_{m-1})}{\Delta_m} + \frac{F(\phi_{m+1})}{\Delta_{m+1}} - \left[\frac{1}{\Delta_m} + \frac{1}{\Delta_{m+1}}\right] F(\phi_m) + \tilde{k}^2 A_\phi(\phi_m)\left[\frac{\Delta_m + \Delta_{m+1}}{2}\right]\right\}$$

$$= a < E_\phi^{inc}, T_m > \equiv V_m \quad \text{(D16)}$$

Moreover, the current is approximated by

$$J_\phi \approx \sum_{i=1}^{N} J_i T_i(\phi) \quad \text{(D17)}$$

to obtain the N x N matrix equation

$$\left\{Z_{mi}\right\} \cdot \left\{J_i\right\} = \left\{V_m\right\} \quad \text{(D18)}$$

from (D16), where

$$Z_{mi} = \frac{j}{\omega\varepsilon\mu a}\left\{\frac{F_{m+1,i}}{\Delta_{m+1}} - F_{m,i}\left[\frac{1}{\Delta_m} + \frac{1}{\Delta_{m+1}}\right] + \frac{F_{m-1,i}}{\Delta_m} + \left[\frac{\Delta_{m+1} + \Delta_m}{2}\right]\tilde{k}^2 A_{mi}\right\} \quad \text{(D19)}$$

$$F_{mi} = \frac{\mu a}{4j} \int T_i(\phi') H_o^{(2)}\left(2ka \sin \frac{|\phi_m - \phi'|}{2}\right) d\phi' \quad (D20)$$

and

$$A_{mi} = \frac{\mu a}{4j} \int T_i(\phi') H_o^{(2)}\left(2ka \sin \frac{|\phi_m - \phi'|}{2}\right) \cos(\phi_m - \phi')\, d\phi' \quad (D21)$$

To fill the matrix Z_{mi}, the singular terms and terms with singular derivatives up to order three in (D20) and (D21) are subtracted out and are integrated analytically. The remaining integrands are evaluated by Gaussian quadrature. Standard techniques are then applied to the linear system (D18) to obtain the coefficients for the currents.

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