> 1
< 0
\( \gamma_1 \):

(5-8)

(5-9)

(5-10)

\( F_x(c) - F_x(-c) \).

\( (x_1) - F_x(x_0) \).

\[ g(x) = g(x_1) = \gamma \quad x_{i-1} < x \leq x_i \]

In this case, the random variable \( y = g(x) \) is of discrete type taking the values \( \gamma_i \) with

\[ P\{y = \gamma_i\} = P\{x_{i-1} < x \leq x_i\} = F_x(x_i) - F_x(x_{i-1}) \]

**Example 5-5**

**Hard Limiter**

If

\[ g(x) = \begin{cases} 
1 & x > 0 \\
-1 & x \leq 0 
\end{cases} \]  

(5-11)

then \( y \) takes the values \( \pm 1 \) with

\[ P\{y = -1\} = P\{x \leq 0\} = F_x(0) \]

\[ P\{y = 1\} = P\{x > 0\} = 1 - F_x(0) \]

Hence \( F_y(y) \) is a staircase function as in Fig. 5-6.
EXAMPLE 5.34 Suppose that $x$ is $N(0; \sigma^2)$ and $y = ax^2$. Inserting into (5-144) and using the evenness of the integrand, we obtain

$$\Phi_y(\omega) = \int_{-\infty}^{\infty} e^{i\omega x^2} f(x) \, dx = \frac{2}{\sigma \sqrt{2\pi}} \int_{0}^{\infty} e^{i\omega x^2} e^{-x^2/2\sigma^2} \, dx$$

As $x$ increases from $0$ to $\infty$, the transformation $y = ax^2$ is one-to-one. Since

$$dy = 2ax \, dx = 2\sqrt{ay} \, dx$$

the last equation yields

$$\Phi_y(\omega) = \frac{2}{\sigma \sqrt{2\pi}} \int_{0}^{\infty} e^{i\omega y} e^{-y/2\sigma^2} \, dy \frac{dy}{2\sqrt{ay}}$$

Hence

$$f_y(y) = \frac{e^{-y/2\sigma^2}}{\sigma \sqrt{2\pi ay}} U(y) \quad (5.145)$$

in agreement with (5.7) and (5.22).

EXAMPLE 5.35 We assume finally that $x$ is uniform in the interval $(-\pi/2, \pi/2)$ and $y = \sin x$. In this case

$$\Phi_y(\omega) = \int_{-\infty}^{\infty} e^{i\omega \sin x} f(x) \, dx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{i\omega \sin x} \, dx$$

As $x$ increases from $-\pi/2$ to $\pi/2$, the function $y = \sin x$ increases from $-1$ to $1$ and

$$dy = \cos x \, dx = \sqrt{1 - y^2} \, dx$$

Hence

$$\Phi_y(\omega) = \frac{1}{\pi} \int_{-1}^{1} e^{i\omega y} \frac{dy}{\sqrt{1 - y^2}}$$

This leads to the conclusion that

$$f_y(y) = \frac{1}{\pi \sqrt{1 - y^2}} \quad \text{for} \quad |y| < 1$$

and $0$ otherwise, in agreement with (5.33).

PROBLEMS

5-1 The random variable $x$ is $N(5, 2)$ and $y = 2x + 4$. Find $E_y$, $\sigma_y$, and $f_y(y)$.

5-2 Find $F_y(y)$ and $f_y(y)$ if $y = -4x + 3$ and $f_x(x) = 2e^{-2x} U(x)$.

5-3 If the random variable $x$ is $N(0, c^2)$ and $g(x)$ is the function in Fig. 5-4, find and sketch the distribution and the density of the random variable $y = g(x)$.

5-4 The random variable $x$ is uniform in the interval $(-2\pi, 2\pi)$. Find and sketch $f_y(y)$ and $F_y(y)$ if $y = g(x)$ and $g(x)$ is the function in Fig. 5-3.

5-5 The random variable $x$ is $N(0, b^2)$ and $g(x)$ is the function in Fig. 5-5. Find and sketch $f_y(y)$ and $F_y(y)$.
The random variable \( x \) is uniform in the interval \((0, 1)\). Find the density of the random variable \( y = -\ln x \).

We place at random 200 points in the interval \((0, 100)\). The distance from 0 to the first random point is a random variable \( z \). Find \( F_z(z) \) exactly and \((b)\) using the Poisson approximation.

If \( y = \sqrt{x} \) and \( x \) is an exponential random variable, show that \( y \) represents a Rayleigh random variable.

Express the density \( f_y(y) \) of the random variable \( y = g(x) \) in terms of \( f_x(x) \) if \((a)\) \( g(x) = |x| \); \((b)\) \( g(x) = e^{-x}U(x) \).

Find \( F_y(y) \) and \( f_y(y) \) if \( F_x(x) = (1 - e^{-2x})U(x) \) and \((a)\) \( y = (x - 1)U(x - 1) \); \((b)\) \( y = x^3 \).

Show that, if the random variable \( x \) has a Cauchy density with \( \alpha = 1 \) and \( y = \arctan x \), then \( y \) is uniform in the interval \((-\pi/2, \pi/2)\).

The random variable \( x \) is uniform in the interval \((-2\pi, 2\pi)\). Find \( f_y(y) \) if \((a)\) \( y = x^3 \), \((b)\) \( y = x^4 \), and \((c)\) \( y = 2\sin(3x + 40^\circ) \).

The random variable \( x \) is uniform in the interval \((-1, 1)\). Find \( g(x) \) such that if \( y = g(x) \), then \( f_y(y) = 2e^{-2y^2}U(y) \).

Given that random variable \( x \) is of continuous type, we form the random variable \( y = g(x) \).

(a) Find \( f_y(y) \) if \( g(x) = 2F_x(x) + 4 \). (b) Find \( g(x) \) such that \( y \) is uniform in the interval \((-8, 10)\).

A fair coin is tossed 10 times and \( x \) equals the number of heads. (a) Find \( F_x(x) \). (b) Find \( F_y(y) \) if \( y = (x - 3)^2 \).

If \( x \) represents a beta random variable with parameters \( \alpha \) and \( \beta \), show that \( 1 - x \) also represents a beta random variable with parameters \( \beta \) and \( \alpha \).

Let \( x \) represent a chi-square random variable with \( n \) degrees of freedom. Then \( y = x^2 \) is known as the chi-distribution with \( n \) degrees of freedom. Determine the p.d.f of \( y \).

Let \( x \sim U(0, 1) \). Show that \( y = -2\log x \) is \( \chi^2(2) \).

If \( x \) is an exponential random variable with parameter \( \lambda \), show that \( y = x^{1/\beta} \) has a Weibull distribution.

If \( t \) is a random variable of continuous type and \( y = a \sin \omega t \), show that

\[
f_y(y) \mathop{\longrightarrow}^{\text{as}} \begin{cases} \frac{1}{\pi} \frac{\sqrt{a^2 - y^2}}{y} & |y| < a \\ 0 & |y| > a \end{cases}
\]

Show that if \( y = x^2 \), then

\[
f_x(x \mid x \geq 0) = \frac{U(y)}{1 - F_x(0)} \frac{f_x(\sqrt{y})}{2\sqrt{y}}
\]

\((a)\) Show that if \( y = \alpha x + b \), then \( \sigma_y = |\alpha| \sigma_x \). \((b)\) Find \( \eta_2 \) and \( \sigma_y \) if \( y = (x - \eta_1)/\sigma_x \).

\((a)\) Show that if \( x \) has a Rayleigh density with parameter \( \alpha \) and \( y = b + cx^2 \), then \( \sigma_y^2 = 4c^2 \sigma_x^4 \).

\((a)\) If \( x \) is \( N(0, 4) \) and \( y = 3x^2 \), find \( \eta_x \), \( \sigma_y \), and \( f_y(y) \).

\((a)\) Let \( x \) represent a binomial random variable with parameters \( n \) and \( p \). Show that \((a)\) \( E(x) = np \); \((b)\) \( E[x(x - 1)] = n(n - 1)p^2 \); \((c)\) \( E[x(x - 1)(x - 2)] = n(n - 1)(n - 2)p^3 \); \((d)\) Compute \( E(x^2) \) and \( E(x^3) \).

For a Poisson random variable \( x \) with parameter \( \lambda \), show that \((a)\) \( P(0 < x < 2\lambda) > (\lambda + 1)/\lambda \); \((b)\) \( E[x(x - 1)] = \lambda^2 \); \((c)\) \( E[x(x - 1)(x - 2)] = \lambda^3 \).

Show that if \( U = \{A_1, \ldots, A_n\} \) is a partition of \( S \), then

\[E(x) = E[x \mid A_1]P(A_1) + \cdots + E[x \mid A_n]P(A_n)\]

\((a)\) Show that if \( x \geq 0 \) and \( E[x] = \eta \), then \( P[x \geq \sqrt{\eta}] \leq \sqrt{\eta} \).

Using (5.86), find \( E[x^2] \) if \( \eta_x = 10 \) and \( \sigma_x = 2 \).
5-30 If \( x \) is uniform in the interval \((10, 12)\) and \( y = x^3 \), (a) find \( f_y(y) \); (b) find \( E[y] \); (i) exactly; (ii) using (4-86).

5-31 The random variable \( x \) is \( N(100, 9) \). Find approximately the mean of the random variable \( y = 1/x \) using (4-86).

5-32 (a) Show that if \( m \) is the median of \( x \), then
\[
E[|x - m|] = E[|x - m|] + 2 \int_a^b (x - m)f(x) \, dx
\]
for any \( a \). (b) Find \( c \) such that \( E[|x - c|] \) is minimum.

5-33 Show that if the random variable \( x \) is \( N(\mu; \sigma^2) \), then
\[
E[|x|] = \sigma \sqrt{\frac{2}{\pi}} e^{-x^2/2\sigma^2} + 2\eta G \left( \frac{\eta}{\sigma} \right) - \eta
\]

5-34 Show that if \( x \) and \( y \) are two random variables with densities \( f_x(x) \) and \( f_y(y) \), respectively, then
\[
E[\log f_x(x)] \geq E[\log f_y(x)]
\]

5-35 (*Chernoff bound*) (a) Show that for any \( \alpha > 0 \) and for any real \( s \),
\[
P(e^{s\alpha} \geq 1) \leq \frac{\Phi(s)}{\alpha}
\]
where \( \Phi(s) = E[e^{\alpha s}] \)

*Hint:* Apply (5-89) to the random variable \( y = e^{s\alpha} \). (b) For any \( A \),
\[
P(x \geq A) \leq e^{-sA} \Phi(s) \quad s > 0
\]
\[
P(x \leq A) \leq e^{-sA} \Phi(s) \quad s < 0
\]

*Hint:* Set \( \alpha = e^{sA} \) in (i).

5-36 Show that for any random variable \( x \)
\[
\left( E[|x|^m]\right)^{1/m} \leq \left( E[|x|^n]\right)^{1/n} \quad 1 < m < n < \infty
\]

5-37 Show that if \( f(x) \) is a Cauchy density, then \( \Phi(\alpha) = e^{-\alpha} \); (b) if \( f(x) \) is a Laplace density, then \( \Phi(\alpha) = e^{-\alpha^2/2} \).

5-38 (a) Let \( x \sim G(\alpha, \beta) \). Show that \( E[x] = \alpha\beta \), \( \text{Var}(x) = \alpha\beta^2 \) and \( \Phi_X(\omega) = (1 - \beta e^{i\omega})^{-\alpha} \).
(b) Let \( x \sim \chi^2(n) \). Show that \( E\{x\} = n, \text{Var}(x) = 2n \) and \( \Phi_X(\omega) = (1 - 2\beta e^{i\omega})^{-n/2} \).
(c) Let \( x \sim B(n, p) \). Show that \( E\{x\} = np, \text{Var}(x) = npq \) and \( \Phi_X(\omega) = (pe^{i\omega} + q)^n \).
(d) Let \( x \sim N(\mu, \sigma) \). Show that \( \Phi_X(\omega) = e^{(1 - qe^{i\omega})/\sigma^2} \).

5-39 A random variable \( x \) has a geometric distribution if
\[
P(x = k) = pq^k \quad k = 0, 1, \ldots \quad p + q = 1
\]
Find \( E(x) \) and show that \( E(x) = q/p, \sigma_x^2 = q/p^2 \).

5-40 Let \( x \) denote the event "the number of failures that precede the \( n \)th success" so that \( x + n \) represents the total number of trials needed to generate \( n \) successes. In that case, the event \( \{x = k\} \) occurs if and only if the last trial results in a success and among the previous \((x + n - 1)\) trials there are \( n - 1 \) successes (or \( x \) failures). This gives an alternate formulation for the Pascal (or negative binomial) distribution as follows: (see Table 5-2)
\[
P(x = k) = \binom{n + k - 1}{k} p^k q^k = \binom{-n}{k} (-q)^k \quad k = 0, 1, 2, \ldots
\]
find \( E(x) \) and show that \( \eta_x = nq/p, \sigma_x^2 = nq/p^2 \).
5.41 Let \( x \) be a negative binomial random variable with parameters \( r \) and \( p \). Show that as \( p \to 1 \) and \( r \to \infty \) such that \( r(1-p) \to \lambda \), a constant, then
\[
P(x = n + r) \to e^{-\lambda} \frac{\lambda^n}{n!} \quad n = 0, 1, 2, \ldots
\]

5.42 Show that if \( E[x] = \eta \), then
\[
E[e^{rx}] = e^{\eta} \sum_{n=0}^{\infty} \frac{\mu_n}{n!} e^{\eta - \eta} = E((x - \eta)^n)
\]

5.43 Show that if \( \Phi, (x) = 1 \) for some \( \omega = 0 \), then the random variable \( x \) is of lattice type taking the values \( x_n = 2n/\omega \).

**Hint:**
\[
0 = 1 - \Phi, (x) = \int_{-\infty}^{\infty} (1 - e^{i\omega x}) f, (x) \, dx
\]

5.44 The random variable \( x \) has zero mean, central moments \( \mu_n \), and cumulants \( \lambda_n \). Show that
\[
\lambda_3 = \mu_3, \lambda_4 = \mu_4 - 3\mu_2^2; \text{ if } y \sim N(\eta; \sigma^2) \text{ and } \sigma_1 = \sigma_2, \text{ then } E(y^4) = E(y^2) + \lambda_3.
\]

5.45 The random variable \( x \) takes the values 0, 1, \ldots with \( P(x = k) = p_k \). Show that if
\[
y = (x - 1)U(x - 1) \quad \text{then} \quad \Gamma, (z) = p_0 + z^{-1} \left[ \Gamma, (z) - p_0 \right]
\]
\[
\eta_x = \eta_x - 1 + p_0 \quad E(y^3) = E(y^2) - 2\eta_x + 1 - p_0
\]

5.46 Show that, if \( \Phi, (w) = E[e^{iwx}] \), then for any \( a_i \),
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \Phi, (w_i - w_j) a_i a_j^* \geq 0
\]

**Hint:**
\[
E \left( \left| \sum_{i=1}^{n} a_i e^{iwx} \right|^2 \right) \geq 0
\]

5.47 We are given an even convex function \( g(x) \) and a random variable \( x \) whose density \( f(x) \) is symmetrical as in Fig. P5-47 with a single maximum at \( x = \eta \). Show that the mean \( E \{ g(x - \eta) \} \) of the random variable \( g(x - \eta) \) is minimum if \( \eta = \eta \).

5.48 The random variable \( x \) is \( N(0; \sigma^2) \). (a) Using characteristic functions, show that if \( g(x) \) is a function such that \( g(x)e^{-x^2/2\sigma^2} \to 0 \) as \( |x| \to \infty \), then (Price's theorem)
\[
\frac{dE[g(x)]}{dv} = \frac{1}{2} E \left\{ \frac{d^2 g(x)}{dx^2} \right\} \quad v = \sigma^2
\]
(b) The moments $\mu_n$ of $x$ are functions of $v$. Using (i), show that

$$\mu_n(v) = \frac{n(n-1)}{2} \int_0^\pi \mu_{n-2}(\beta) \, d\beta$$

5-49 Show that, if $x$ is an integer-valued random variable with moment function $\Gamma(z)$ as in (5-113), then

$$P(x = k) = \frac{1}{2\pi} \int_{-\pi}^\pi \Gamma(e^{i\omega}) e^{-ik\omega} \, d\omega$$

5-50 A biased coin is tossed and the first outcome is noted. The tossing is continued until the outcome is the complement of the first outcome, thus completing the first run. Let $x$ denote the length of the first run. Find the p.m.f of $x$, and show that

$$E(x) = \frac{p}{q} + \frac{q}{p}$$

5-51 A box contains $N$ identical items of which $M < N$ are defective ones. A sample of size $n$ is taken from the box, and let $x$ represent the number of defective items in this sample. (a) Find the distribution function of $x$ if the $n$ samples are drawn with replacement. (b) If the $n$ samples are drawn without replacement, then show that

$$P(x = k) = \binom{M}{k} \binom{N-M}{n-k} \binom{N}{n} \max(0, n + M - N) \leq k \leq \min(M, N)$$

Find the mean and variance of $x$. The distribution in (b) is known as the hypergeometric distribution (see also Problem 3-5). The lottery distribution in (3-39) is an example of this distribution. (c) In (b), let $N \to \infty$, $M \to \infty$, such that $M/N \to p$, $0 < p < 1$. Then show that the hypergeometric random variable can be approximated by a Binomial random variable with parameters $n$ and $p$, provided $n \ll N$.

5-52 A box contains $n$ white and $m$ black marbles. Let $x$ represent the number of draws needed for the $r$th white marble. (a) If sampling is done with replacement, show that $x$ has a negative binomial distribution with parameters $r$ and $p = n/(n + m)$. (b) If sampling is done without replacement, then show that

$$P(x = k) = \binom{k-1}{r-1} \binom{m-n}{w-r} \binom{n+m}{w} \quad k = r, r + 1, \ldots, m + n$$

(c) For a given $k$ and $r$, show that the probability distribution in (b) tends to a negative binomial distribution as $n + m \to \infty$. Thus, for large population size, sampling with or without replacement is the same.