1. \[ \begin{align*}
\Phi_1 & = A z + B \\
\Phi_2 & = C z + D \\
\Phi(z=0) & = V_0 \\
\Phi(z=a+b) & = 0 \\
\Phi_1(z=a) & = \Phi_2(z=a+b)
\end{align*} \]

a) Know \( \nabla^2 \Phi = 0 \) in each homogeneous region.

Because problem is independent of \( x, y \)

Have \( \frac{\partial^2 \Phi_1}{\partial z^2} = 0 \) in region I

\( \frac{\partial^2 \Phi_2}{\partial z^2} = 0 \) in region II

Means

\( \Phi_1 = A z + B \)
\( \Phi_2 = C z + D \)

Know

\( \Phi_1(z=0) = \Phi_2(z=a+b) = \Phi(z=a+b) = 0 \)

\( D = -C(z=a+b) \)

Also know: \( \Phi \) should be continuous across the interface.

Also know that \( D_{2n} - D_{1n} = \rho_s \), but no free charge

At interface \( \Rightarrow \rho_s = 0 \) for \( z = a \).

Then \( D_{2n} = D_{1n} \) at \( z = a \).

\[ \Rightarrow \varepsilon_{2n} E_{2n} = \varepsilon_{1n} E_{1n} \]

But \( \hat{n} \cdot \hat{z} \) and \( E_{1n} = -\hat{n} \cdot \nabla \Phi \)

\[ \begin{align*}
E_{2n} & = -\frac{\partial}{\partial z} \Phi_2 = -C \\
E_{1n} & = -\frac{\partial}{\partial z} \Phi_1 = -A
\end{align*} \]
Thus \( e_2 E_{zn} = e_1 E_{in} \) means \( e_2 C = e_1 A \)
and
\[
\Phi_1 (z = a) = a \Phi_2 (z = a) \text{ means } A_0 + B = C_0 + D
\]
Therefore we have 4 eqns for 4 unknowns
\[
B = \nabla_0 \\
A a + B = C a + D \\
e_2 C = e_1 A \\
D = -C (a + b) = -\frac{e_1}{e_2} (a + b) A
\]
\[
A a + \nabla_0 = \frac{e_1 A a}{e_2} - \frac{e_1}{e_2} (a + b) A = -\frac{e_1}{e_2} b A
\]
\[
\Rightarrow \ A (a + \frac{e_1}{e_2} b) = -\nabla_0 \quad A = -\frac{e_2 \nabla_0}{e_2 a + e_1 b}
\]
Thus
\[
\Phi_1 (z) = \nabla_0 - \frac{e_2 \nabla_0}{e_2 a + e_1 b} z
\]
\[
\Phi_2 (z) = C \left[ z - (a + b) \right]
\]
\[
= \frac{e_1 \nabla_0}{e_2 a + e_1 b} (a + b - z)
\]
Checks:
\[
\Phi_1 (z = 0) = \nabla_0 \quad \Phi_2 (z = a + b) = 0
\]
\[
\Phi_1 (z = a) = \nabla_0 \left[ 1 - \frac{e_2 a}{e_2 a + e_1 b} \right] = \frac{\nabla_0 e_1 b}{e_2 a + e_1 b}
\]
\[
\Phi_2 (z = a) = \frac{e_1 b \nabla_0}{e_2 a + e_1 b} = \Phi_1 (z = a)
\]
\[
- e_1 \frac{\partial}{\partial z} \Phi_1 (z = a) = e_2 e_1 \nabla_0 / (e_2 a + e_1 b)
\]
\[-e_2 \frac{\partial}{\partial z} \Phi_2 (z=a) = \frac{\epsilon_2 \epsilon_1 V_0}{\epsilon_2 \epsilon a + \epsilon_1 b} = -e_1 \frac{\partial}{\partial z} \Phi_1 (z=1)\]

The solution checks out!

b) \[\vec{E}_1 (z) = -\frac{\partial}{\partial z} \Phi_1 \hat{\imath} = \frac{\epsilon_2 V_0}{\epsilon_2 \epsilon a + \epsilon_1 b} \hat{\imath}\]

\[\vec{E}_2 (z) = -\frac{\partial}{\partial z} \hat{\Phi}_2 \hat{\imath} = \frac{\epsilon_1 V_0}{\epsilon_2 \epsilon a + \epsilon_1 b} \hat{\imath}\]

c) There is a physical discontinuity because of the different number of dipoles in each medium. We know \[\vec{D} = \epsilon_0 \vec{E} + \vec{P}\]
and \[\vec{D}_n = \vec{D}_1 + \vec{D}_2\] at the dielectric interface.

This means
\[\epsilon_0 E_{1z} (z=a) + P_{1z} (z=a) = \epsilon_0 E_{2z} (z=a) + P_{2z} (z=1)\]

\[\Rightarrow P_{2z} (z=a) = P_{1z} (z=a) + \epsilon_0 \left[ E_{1z} (z=a) - E_{2z} (z=a) \right]\]

\[= P_{1z} (z=a) + \frac{\epsilon_0 V_0}{\epsilon_2 \epsilon a + \epsilon_1 b} (\epsilon_2 - \epsilon_1)\]

\[\Rightarrow\] polarization fields are discontinuous at \(z = a\).

\[\epsilon_0 \vec{E} (x=0, y=0, z=0) = \sum_{l=1}^{4} \frac{Q_l}{4\pi R_l}\]

\[R_1 = R_2 = R_3 = R_4 = 12\]

\[Q_1 + Q_2 + Q_3 + Q_4 = 0\]

\[\Rightarrow \vec{E} (x=0, y=0, z=0) \equiv 0\]
Electric Field at origin = 0.

\[ E = \frac{\lambda}{2\pi\epsilon_0} \left[ x(1-1-1-1) + \frac{\lambda}{2} (1+1+1+1) \right] \]

\[ E = \frac{\lambda}{2} \frac{\epsilon_0}{2\pi} \]

\[ E(x=0, y=0, z=0) = \]

\[ \sum \frac{\lambda}{2\pi\epsilon_0} \left( -x \right) \]

\[ R_1 = R_2 = R_3 = R_4 = x = z = 0 \]

\[ \frac{R_4}{R_1} = -x \]

\[ \frac{R_4}{R_1} = \frac{x^2}{\lambda^2} \]

\[ R_1 = \frac{x^2}{\lambda^2} \]

\[ E = \frac{\lambda}{2\pi\epsilon_0} \left( -x \right) \]

\[ \sum \frac{\lambda}{2\pi\epsilon_0} \left( -x \right) = \]

\[ \frac{R_4}{R_1} = -x \]

\[ \frac{R_4}{R_1} = \frac{x^2}{\lambda^2} \]

\[ R_1 = \frac{x^2}{\lambda^2} \]

\[ E = \sum \frac{\lambda}{2\pi\epsilon_0} \left( -x \right) \]

\[ \frac{R_4}{R_1} = -x \]

\[ \frac{R_4}{R_1} = \frac{x^2}{\lambda^2} \]

\[ R_1 = \frac{x^2}{\lambda^2} \]
3. **Coulomb Law Problem**

\[ \oint_S \vec{E} \cdot \hat{n}_{S} \, dS = Q_{\text{inside } S} \]

a. For spheres \( \hat{n}_S = \hat{R} \)

\[ dS = R^2 \sin \theta \, d\theta \, d\phi \]

\[ \oint_S \vec{D} \cdot \hat{n}_{S} \, dS = 4\pi R^2 \vec{E}(R) \]

\[ \vec{E}(r, \theta, \phi) = \vec{E}(R) \hat{R} \]

i) for \( R < a \)

\[ Q_{\text{inside } S} = 0 \Rightarrow E(r, \theta, \phi) = 0 \]

\[ \Rightarrow E(r, \theta, \phi) = 0 \quad \text{for } R < a \]

ii) for \( R = a \)

\[ Q_{\text{inside } S} = Q \]

\[ \Rightarrow E_r(R) = \frac{Q}{4\pi \varepsilon} \frac{1}{R^2} \]

\[ \Rightarrow \vec{E}(r, \theta, \phi) = \frac{Q}{4\pi \varepsilon} \frac{1}{R^2} \hat{R} \] (labeled change)

b. \[ \left| \frac{Q}{4\pi \varepsilon a^2} \right| \alpha \frac{1}{R^2} \]

\[ R=0 \quad \rightarrow \quad R=a \]

\[ \vec{E} \]

\[ \vec{E}(R) \]

\[ \vec{E} \cdot d\vec{a} \]

\[ \int_{A} \vec{E} \cdot d\vec{a} \]

\[ \Omega (\bar{B}) - \Omega (\bar{A}) = - \int_{\bar{A}} \vec{E} \cdot d\vec{a} \]

\[ \Rightarrow \vec{V}(\bar{B}) = \vec{V}(\bar{A}) \]

\[ \text{i) for } R < a: \quad \vec{E} = 0 \Rightarrow \]

\[ \text{For any two points } \bar{A}, \bar{B} \text{ inside sphere } R=a, \quad \vec{V}(\bar{A}) = \vec{V}(\bar{B}) \]
(ii) For $R > a$ let $A = \text{infinity}$, $B = R^2$

$$V(R) - V(\infty) = - \int_{\infty}^{R} \left( \frac{Q}{4\pi \epsilon R^2} \right) \cdot \left( \frac{1}{R'} \right) \, dR'$$

$$= + \frac{Q}{4\pi \epsilon} \left. \frac{1}{R} \right|_{\infty}^{R} = \frac{Q}{4\pi \epsilon} \frac{1}{R}$$

But $V(\infty) = 0$ for finite source.

Thus

$$V(R) = \frac{Q}{4\pi \epsilon} \frac{1}{R}$$

(like pt charge)

Also

$$V(R = a, \theta, \phi) = \frac{Q}{4\pi \epsilon} \frac{1}{a}$$

Because $V$ must be continuous at $R = a$,

then

$$V(R, \theta, \phi) = \frac{Q}{4\pi \epsilon} \frac{1}{a}$$

for $R < a$

---

d) \[ V \]

\[ \frac{Q}{4\pi \epsilon a} \]

\[ R=0 \quad R=a \quad R \]

\[ a \frac{1}{R} \]

Notes: Derivative of $V = \frac{Q}{4\pi \epsilon a}$ = constant for $R < a$ gives $E = 0$ for $R < a$. 
Problem 4.24 Charge $Q_1$ is uniformly distributed over a thin spherical shell of radius $a$, and charge $Q_2$ is uniformly distributed over a second spherical shell of radius $b$, with $b > a$. Apply Gauss’s law to find $E$ in the regions $R < a$, $a < R < b$, and $R > b$.

Solution: Using symmetry considerations, we know $D = \hat{R}D_R$. From Table 3.1, $ds = \hat{R}R^2\sin\theta \, d\theta \, d\phi$ for an element of a spherical surface. Using Gauss’s law in integral form (Eq. (4.29)),

$$\int_S D \cdot ds = Q_{\text{tot}},$$

where $Q_{\text{tot}}$ is the total charge enclosed in $S$. For a spherical surface of radius $R$,

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} (\hat{R}D_R) \cdot (\hat{R}R^2\sin\theta \, d\theta \, d\phi) = Q_{\text{tot}},$$

$$D_RR^2(2\pi)[-\cos\theta]_0^{\pi} = Q_{\text{tot}},$$

$$D_R = \frac{Q_{\text{tot}}}{4\pi R^2}.$$

From Eq. (4.15), we know a linear, isotropic material has the constitutive relationship $D = \varepsilon E$. Thus, we find $E$ from $D$.

(a) In the region $R < a$,

$$Q_{\text{tot}} = 0, \quad E = \hat{R}E_R = \frac{\hat{R}Q_{\text{tot}}}{4\pi R^2 \varepsilon} = 0 \text{ (V/m)}.$$

(b) In the region $a < R < b$,

$$Q_{\text{tot}} = Q_1, \quad E = \hat{R}E_R = \frac{\hat{R}Q_1}{4\pi R^2 \varepsilon} \text{ (V/m)}.$$

(c) In the region $R > b$,

$$Q_{\text{tot}} = Q_1 + Q_2, \quad E = \hat{R}E_R = \frac{\hat{R}(Q_1 + Q_2)}{4\pi R^2 \varepsilon} \text{ (V/m)}.$$
**Problem 4.37** Two infinite lines of charge, both parallel to the $z$-axis, lie in the $x$–$z$ plane, one with density $\rho_l$ and located at $x = a$ and the other with density $-\rho_l$ and located at $x = -a$. Obtain an expression for the electric potential $V(x, y)$ at a point $P = (x, y)$ relative to the potential at the origin.

**Solution:** According to the result of Problem 4.33, the electric potential difference between a point at a distance $r_1$ and another at a distance $r_2$ from a line charge of density $\rho_l$ is

$$V = \frac{\rho_l}{2\pi\varepsilon_0} \ln \left( \frac{r_2}{r_1} \right).$$

Applying this result to the line charge at $x = a$, which is at a distance $a$ from the origin:

$$V' = \frac{\rho_l}{2\pi\varepsilon_0} \ln \left( \frac{a}{r'} \right) \quad (r_2 = a \text{ and } r_1 = r')$$

$$= \frac{\rho_l}{2\pi\varepsilon_0} \ln \left( \frac{a}{\sqrt{(x-a)^2 + y^2}} \right).$$

Similarly, for the negative line charge at $x = -a$,

$$V'' = -\frac{\rho_l}{2\pi\varepsilon_0} \ln \left( \frac{a}{r''} \right) \quad (r_2 = a \text{ and } r_1 = r')$$

$$= -\frac{\rho_l}{2\pi\varepsilon_0} \ln \left( \frac{a}{\sqrt{(x+a)^2 + y^2}} \right).$$

The potential due to both lines is

$$V = V' + V'' = \frac{\rho_l}{2\pi\varepsilon_0} \left[ \ln \left( \frac{a}{\sqrt{(x-a)^2 + y^2}} \right) - \ln \left( \frac{a}{\sqrt{(x+a)^2 + y^2}} \right) \right].$$

At the origin, $V = 0$, as it should be since the origin is the reference point. The potential is also zero along all points on the $y$-axis ($x = 0$).
**Problem 4.40**  The $x$–$y$ plane contains a uniform sheet of charge with $\rho_{s_1} = 0.2$ (nC/m$^2$). A second sheet with $\rho_{s_2} = -0.2$ (nC/m$^2$) occupies the plane $z = 6$ m. Find $V_{AB}$, $V_{BC}$, and $V_{AC}$ for $A(0, 0, 6)$, $B(0, 0, 0)$, and $C(0, -2, 2)$.

**Solution:** We start by finding the $E$ field in the region between the plates. For any point above the $x$–$y$ plane, $E_1$ due to the charge on $x$–$y$ plane is, from Eq. (4.25),

$$E_1 = \hat{z} \frac{\rho_{s_1}}{2\varepsilon_0}.$$ 

In the region below the top plate, $E$ would point downwards for positive $\rho_{s_2}$ on the top plate. In this case, $\rho_{s_2} = -\rho_{s_1}$. Hence,

$$E = E_1 + E_2 = \hat{z} \frac{\rho_{s_1}}{2\varepsilon_0} - \hat{z} \frac{2\rho_{s_1}}{2\varepsilon_0} = \hat{z} \frac{\rho_{s_1}}{2\varepsilon_0} = \hat{z} \frac{\rho_{s_1}}{\varepsilon_0}.$$ 

Since $E$ is along $\hat{z}$, only change in position along $z$ can result in change in voltage.

$$V_{AB} = -\int_0^6 \hat{z} \frac{\rho_{s_1}}{\varepsilon_0} \; dz = -\frac{\rho_{s_1}}{\varepsilon_0} \int_0^6 \hat{z} \; dz = -\frac{6 \rho_{s_1}}{\varepsilon_0} = -\frac{6 \times 0.2 \times 10^{-9}}{8.85 \times 10^{-12}} = -135.59 \text{ V.}$$

The voltage at $C$ depends only on the $z$-coordinate of $C$. Hence, with point $A$ being at the lowest potential and $B$ at the highest potential,

$$V_{BC} = -\frac{2}{6} V_{AB} = -\frac{(-135.59)}{3} = 45.20 \text{ V},$$

$$V_{AC} = V_{AB} + V_{BC} = -135.59 + 45.20 = -90.39 \text{ V.}$$
Figure P4.40: Two parallel planes of charge.
**Problem 4.15** Electric charge is distributed along an arc located in the $x$–$y$ plane and defined by $r = 2$ cm and $0 \leq \phi \leq \pi/4$. If $\rho_\ell = 5$ ($\mu$C/m), find $E$ at $(0,0,z)$ and then evaluate it at:

- (a) The origin.
- (b) $z = 5$ cm
- (c) $z = -5$ cm

**Solution:** For the arc of charge shown in Fig. P4.15, $dl = r \, d\phi = 0.02 \, d\phi$, and $R' = -\hat{x}0.02 \cos \phi - \hat{y}0.02 \sin \phi + \hat{z}z$. Use of Eq. (4.21c) gives

$$E = \frac{1}{4\pi \varepsilon_0} \int_{\phi=0}^{\pi/4} \frac{R' \rho_\ell \, dl'}{R'^2}$$

$$= \frac{0.02}{4\pi \varepsilon_0} \int_{\phi=0}^{\pi/4} \left( -\hat{x}0.02 \cos \phi - \hat{y}0.02 \sin \phi + \hat{z}z \right) \frac{0.02 \, d\phi}{((0.02)^2 + z^2)^{3/2}}$$

$$= \frac{898.8}{((0.02)^2 + z^2)^{3/2}} \left[ -\hat{x}0.014 - \hat{y}0.006 + \hat{z}0.78z \right] \text{(V/m)}.$$

- (a) At $z = 0$, $E = -\hat{x}1.6 - \hat{y}0.06$ (MV/m).
- (b) At $z = 5$ cm, $E = -\hat{x}81.4 - \hat{y}33.7 + \hat{z}226$ (kV/m).
- (c) At $z = -5$ cm, $E = -\hat{x}81.4 - \hat{y}33.7 - \hat{z}226$ (kV/m).

![Figure P4.15: Line charge along an arc.](image-url)
Problem 4.25  The electric flux density inside a dielectric sphere of radius $a$ centered at the origin is given by

$$\mathbf{D} = \hat{\mathbf{R}} \rho_0 R \quad (\text{C/m}^2)$$

where $\rho_0$ is a constant. Find the total charge inside the sphere.

Solution:

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{s} = \int_0^\pi \int_0^{2\pi} \hat{\mathbf{R}} \rho_0 R \cdot \hat{\mathbf{R}} R^2 \sin \theta \ d\theta \ d\phi \bigg|_{R=a}$$

$$= 2\pi \rho_0 a^3 \int_0^\pi \sin \theta \ d\theta = -2\pi \rho_0 a^3 \cos \theta \bigg|_0^\pi = 4\pi \rho_0 a^3 \quad (\text{C}).$$
Problem 4.30 A square in the \( x-y \) plane in free space has a point charge of \( +Q \) at corner \( (a/2, a/2) \), the same at corner \( (a/2, -a/2) \), and a point charge of \( -Q \) at each of the other two corners.

(a) Find the electric potential at any point \( P \) along the \( x \)-axis.

(b) Evaluate \( V \) at \( x = a/2 \).

Solution: \( R_1 = R_2 \) and \( R_3 = R_4 \).

![Figure P4.30: Potential due to four point charges.](image)

\[
V = \frac{Q}{4\pi \varepsilon_0 R_1} + \frac{Q}{4\pi \varepsilon_0 R_2} + \frac{-Q}{4\pi \varepsilon_0 R_3} + \frac{-Q}{4\pi \varepsilon_0 R_4} = \frac{Q}{2\pi \varepsilon_0} \left( \frac{1}{R_1} - \frac{1}{R_3} \right)
\]

with

\[
R_1 = \sqrt{\left( x - \frac{a}{2} \right)^2 + \left( \frac{a}{2} \right)^2},
\]

\[
R_3 = \sqrt{\left( x + \frac{a}{2} \right)^2 + \left( \frac{a}{2} \right)^2}.
\]

At \( x = a/2 \),

\[
R_1 = \frac{a}{2}.
\]
\[ R_3 = \frac{a\sqrt{5}}{2}, \]
\[ V = \frac{Q}{2\pi\varepsilon_0} \left( \frac{2}{a} - \frac{2}{\sqrt{5}a} \right) = \frac{0.55Q}{\pi\varepsilon_0 a}. \]
**Problem 4.31** The circular disk of radius $a$ shown in Fig. 4-7 has uniform charge density $\rho_s$ across its surface.

(a) Obtain an expression for the electric potential $V$ at a point $P = (0,0,z)$ on the $z$-axis.

(b) Use your result to find $E$ and then evaluate it for $z = h$. Compare your final expression with (4.24), which was obtained on the basis of Coulomb’s law.

**Solution:**

![Figure P4.31: Circular disk of charge.](image)

(a) Consider a ring of charge at a radial distance $r$. The charge contained in width $dr$ is

$$dq = \rho_s (2\pi r \, dr) = 2\pi \rho_s r \, dr.$$  

The potential at $P$ is

$$dV = \frac{dq}{4\pi\varepsilon_0 R} = \frac{2\pi \rho_s r \, dr}{4\pi\varepsilon_0 (r^2 + z^2)^{1/2}}.$$  

The potential due to the entire disk is

$$V = \int_0^a dV = \frac{\rho_s}{2\varepsilon_0} \int_0^a \frac{r \, dr}{(r^2 + z^2)^{1/2}} = \frac{\rho_s}{2\varepsilon_0} \left[ \frac{(a^2 + z^2)^{1/2}}{2} \right]_0^a = \frac{\rho_s}{2\varepsilon_0} \left[ (a^2 + z^2)^{1/2} - z \right].$$
(b) \[
E = -\nabla V = -\hat{x} \frac{\partial V}{\partial x} - \hat{y} \frac{\partial V}{\partial y} - \hat{z} \frac{\partial V}{\partial z} = \hat{z} \frac{\rho_s}{2 \varepsilon_0} \left[ 1 - \frac{z}{\sqrt{a^2 + z^2}} \right].
\]

The expression for \( E \) reduces to Eq. (4.24) when \( z = h \).
**Problem 4.38**  Given the electric field

\[ \mathbf{E} = \hat{\mathbf{r}} \frac{18}{R^2} \text{ (V/m)} \]

find the electric potential of point \( A \) with respect to point \( B \) where \( A \) is at \(+2\) m and \( B \) at \(-4\) m, both on the \( z\)-axis.

**Solution:**

\[ V_{AB} = V_A - V_B = -\int_B^A \mathbf{E} \cdot d\mathbf{l}. \]

Along \( z\)-direction, \( \hat{\mathbf{r}} = \hat{\mathbf{z}} \) and \( \mathbf{E} = \hat{\mathbf{z}} \frac{18}{z^2} \) for \( z \geq 0 \), and \( \hat{\mathbf{r}} = -\hat{\mathbf{z}} \) and \( \mathbf{E} = -\hat{\mathbf{z}} \frac{18}{z^2} \) for \( z \leq 0 \). Hence,

\[ V_{AB} = -\int_{-4}^{2} \frac{18}{z^2} \hat{\mathbf{z}} \cdot d\mathbf{z} = 4 \text{ V}. \]