Problem 8.18 For some types of glass, the index of refraction varies with wavelength. A prism made of a material with

\[ n = 1.71 - \frac{4}{30} \lambda_0 \]  
(\( \lambda_0 \) in \( \mu m \)),

where \( \lambda_0 \) is the wavelength in vacuum, was used to disperse white light as shown in Fig. P8.18. The white light is incident at an angle of 50°, the wavelength \( \lambda_0 \) of red light is 0.7 \( \mu m \), and that of violet light is 0.4 \( \mu m \). Determine the angular dispersion in degrees.

Solution:

For violet,

\[ n_v = 1.71 - \frac{4}{30} \times 0.4 = 1.66, \quad \sin \theta_2 = \frac{\sin \theta}{n_v} = \frac{\sin 50^\circ}{1.66}, \]

or

\[ \theta_2 = 27.48^\circ. \]

From the geometry of triangle \( ABC \),

\[ 180^\circ = 60^\circ + (90^\circ - \theta_2) + (90^\circ - \theta_3), \]

or

\[ \theta_3 = 60^\circ - \theta_2 = 60 - 27.48^\circ = 32.52^\circ, \]

and

\[ \sin \theta_4 = n_v \sin \theta_3 = 1.66 \sin 32.52^\circ = 0.89, \]

or

\[ \theta_4 = 63.18^\circ. \]
For red,

\[ n_r = 1.71 - \frac{4}{30} \times 0.7 = 1.62, \]

\[ \theta_2 = \sin^{-1} \left[ \frac{\sin 50^\circ}{1.62} \right] = 28.22^\circ, \]

\[ \theta_3 = 60^\circ - 28.22^\circ = 31.78^\circ, \]

\[ \theta_4 = \sin^{-1} [1.62 \sin 31.78^\circ] = 58.56^\circ. \]

Hence, angular dispersion = 63.18° − 58.56° = 4.62°.
Problem 8.27  A plane wave in air with
\[ \mathbf{E}^i = \hat{y} \times 20e^{-j(3x+4z)} \text{ (V/m)} \]
is incident upon the planar surface of a dielectric material, with \( \varepsilon_r = 4 \), occupying the half-space \( z \geq 0 \). Determine:

(a) The polarization of the incident wave.
(b) The angle of incidence.
(c) The time-domain expressions for the reflected electric and magnetic fields.
(d) The time-domain expressions for the transmitted electric and magnetic fields.
(e) The average power density carried by the wave in the dielectric medium.

Solution:
(a) \( \mathbf{E}^i = \hat{y} \times 20e^{-j(3x+4z)} \text{ V/m} \).

Since \( \mathbf{E}^i \) is along \( \hat{y} \), which is perpendicular to the plane of incidence, the wave is perpendicularly polarized.

(b) From Eq. (8.48a), the argument of the exponential is
\[-jk_1(x \sin \theta_i + z \cos \theta_i) = -j(3x + 4z) .\]

Hence,
\[ k_1 \sin \theta_i = 3, \quad k_1 \cos \theta_i = 4, \]
from which we determine that
\[ \tan \theta_i = \frac{3}{4} \quad \text{or} \quad \theta_i = 36.87^\circ, \]
and
\[ k_1 = \sqrt{3^2 + 4^2} = 5 \text{ (rad/m)}. \]

Also,
\[ \omega = \mu_0 k = ck = 3 \times 10^8 \times 5 = 1.5 \times 10^9 \text{ (rad/s)}. \]

(c)
\[ \eta_1 = \eta_0 = 377 \text{ } \Omega, \]
\[ \eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{\eta_0}{2} = 188.5 \text{ } \Omega, \]
\[ \theta_t = \sin^{-1} \left[ \frac{\sin \theta_i}{\sqrt{\varepsilon_r}} \right] = \sin^{-1} \left[ \frac{\sin 36.87^\circ}{\sqrt{4}} \right] = 17.46^\circ, \]
\[ \Gamma_{\perp} = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} = -0.41, \]
\[ \tau_\perp = 1 + \Gamma_\perp = 0.59. \]

In accordance with Eq. (8.49a), and using the relation \( E^r_0 = \Gamma_\perp E^i_0 \),

\[
\begin{align*}
\vec{E}^r &= -\hat{y} \cdot 8.2 \cdot e^{-j(3x - 4z)} , \\
\vec{H}^r &= -(\hat{x} \cos \theta_i + \hat{z} \sin \theta_i) \cdot \frac{8.2}{\eta_0} \cdot e^{-j(3x - 4z)},
\end{align*}
\]

where we used the fact that \( \theta_i = \theta_r \) and the \( z \)-direction has been reversed.

\[
\begin{align*}
E^r &= \Re\{\vec{E}^r e^{j\omega t}\} = -\hat{y} \cdot 8.2 \cdot \cos(1.5 \times 10^9 t - 3x + 4z) \quad \text{(V/m)}, \\
H^r &= -(\hat{x} \cdot 17.4 + \hat{z} \cdot 13.06) \cdot \cos(1.5 \times 10^9 t - 3x + 4z) \quad \text{(mA/m)}.
\end{align*}
\]

\textbf{(d)} In medium 2,

\[
k_2 = k_1 \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = 5 \sqrt{4} = 20 \quad \text{(rad/m)},
\]

and

\[
\theta_i = \sin^{-1}\left[ \left( \frac{\varepsilon_1}{\varepsilon_2} \right) \sin \theta_1 \right] = \sin^{-1}\left[ \frac{1}{2} \sin 36.87^\circ \right] = 17.46^\circ
\]

and the exponent of \( E^r \) and \( H^r \) is

\[-jk_2(x \sin \theta_i + z \cos \theta_i) = -j10(x \sin 17.46^\circ + z \cos 17.46^\circ) = -j(3x + 9.54z).\]

Hence,

\[
\begin{align*}
\vec{E}^r &= \hat{y} \cdot 20 \times 0.59 \cdot e^{-j(3x + 9.54z)}, \\
\vec{H}^r &= -(\hat{x} \cos \theta_i + \hat{z} \sin \theta_i) \cdot \frac{20 \times 0.59}{\eta_2} \cdot e^{-j(3x + 9.54z)}. \\
E^r &= \Re\{\vec{E}^r e^{j\omega t}\} = \hat{y} \cdot 11.8 \cos(1.5 \times 10^9 t - 3x - 9.54z) \quad \text{(V/m)}, \\
H^r &= -(\hat{x} \cos 17.46^\circ + \hat{z} \sin 17.46^\circ) \cdot \frac{11.8}{188.5} \cos(1.5 \times 10^9 t - 3x - 9.54z) \\
&= -(\hat{x} \cdot 59.72 + \hat{z} \cdot 18.78) \cdot \cos(1.5 \times 10^9 t - 3x - 9.54z) \quad \text{(mA/m)}.
\end{align*}
\]

\textbf{(e)}

\[
S_{aw} = \frac{|E^r_0|^2}{2\eta_2} = \frac{(11.8)^2}{2 \times 188.5} = 0.36 \quad \text{(W/m²)}.
\]
Problem 9.2  A 1-m–long dipole is excited by a 1-MHz current with an amplitude of 12 A. What is the average power density radiated by the dipole at a distance of 5 km in a direction that is $45^\circ$ from the dipole axis?

Solution: At 1 MHz, $\lambda = c/f = 3 \times 10^8 / 10^6 = 300$ m. Hence $l/\lambda = 1/300$, and therefore the antenna is a Hertzian dipole. From Eq. (9.12),

$$S(R, \theta) = \left( \frac{\eta_0 k^2 I_0^2 l^2}{32 \pi^2 R^2} \right) \sin^2 \theta$$

$$= \frac{120\pi \times (2\pi/300)^2 \times 12^2 \times 1^2}{32 \pi^2 \times (5 \times 10^3)^2} \sin^2 45^\circ = 1.51 \times 10^{-9} \text{ (W/m}^2)\text{).}$$
**Problem 9.12**  Assuming the loss resistance of a half-wave dipole antenna to be negligibly small and ignoring the reactance component of its antenna impedance, calculate the standing-wave ratio on a 50-Ω transmission line connected to the dipole antenna.

**Solution:**  According to Eq. (9.48), a half wave dipole has a radiation resistance of 73 Ω. To the transmission line, this behaves as a load, so the reflection coefficient is

\[
\Gamma = \frac{R_{\text{rad}} - Z_0}{R_{\text{rad}} + Z_0} = \frac{73 \ \Omega - 50 \ \Omega}{73 \ \Omega + 50 \ \Omega} = 0.187,
\]

and the standing wave ratio is

\[
S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.187}{1 - 0.187} = 1.46.
\]
Problem 9.20  A 3-GHz line-of-sight microwave communication link consists of two lossless parabolic dish antennas, each 1 m in diameter. If the receive antenna requires 10 nW of receive power for good reception and the distance between the antennas is 40 km, how much power should be transmitted?

Solution: At $f = 3$ GHz, $\lambda = c/f = (3 \times 10^8 \text{ m/s})/(3 \times 10^9 \text{ Hz}) = 0.10 \text{ m}$. Solving the Friis transmission formula (Eq. (9.75)) for the transmitted power:

$$P_t = P_{\text{rec}} \frac{\lambda^2 R^2}{4 \pi \xi_s \xi_r A_t A_r}$$

$$= 10^{-8} \frac{(0.100 \text{ m})^2 (40 \times 10^3 \text{ m})^2}{1 \times 1 \times (\frac{\pi}{4} (1 \text{ m})^2)(\frac{\pi}{4} (1 \text{ m})^2)} = 25.9 \times 10^{-2} \text{ W} = 259 \text{ mW}.$$
Problem 8.19  The two prisms in Fig. P8.19 are made of glass with \( n = 1.5 \). What fraction of the power density carried by the ray incident upon the top prism emerges from the bottom prism? Neglect multiple internal reflections.

![Periscope prisms of Problem 8.19.](image)

**Figure P8.19:** Periscope prisms of Problem 8.19.

**Solution:** Using \( \eta = \eta_0/n \), at interfaces 1 and 4,

\[
\Gamma_a = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1 - 1.5}{1 + 1.5} = -0.2.
\]

At interfaces 3 and 6,

\[
\Gamma_b = -\Gamma_a = 0.2.
\]

At interfaces 2 and 5,

\[
\theta_c = \sin^{-1}\left(\frac{1}{n}\right) = \sin^{-1}\left(\frac{1}{1.5}\right) = 41.81^\circ.
\]

Hence, total internal reflection takes place at those interfaces. At interfaces 1, 3, 4 and 6, the ratio of power density transmitted to that incident is \((1 - \Gamma^2)\). Hence,

\[
\frac{S_f}{S_i} = (1 - \Gamma^2)^4 = (1 - (0.2)^2)^4 = 0.85.
\]
**Problem 8.22**  Figure P8.22 depicts a beaker containing a block of glass on the bottom and water over it. The glass block contains a small air bubble at an unknown depth below the water surface. When viewed from above at an angle of $60^\circ$, the air bubble appears at a depth of 6.81 cm. What is the true depth of the air bubble?

**Solution:** Let 

$$d_a = 6.81 \text{ cm} = \text{apparent depth},$$

$$d_t = \text{true depth}.$$ 

$$\theta_2 = \sin^{-1} \left( \frac{n_1}{n_2} \sin \theta_1 \right) = \sin^{-1} \left( \frac{1}{1.33} \sin 60^\circ \right) = 40.6^\circ,$$

$$\theta_3 = \sin^{-1} \left( \frac{n_1}{n_3} \sin \theta_1 \right) = \sin^{-1} \left( \frac{1}{1.6} \sin 60^\circ \right) = 32.77^\circ,$$

$$x_1 = (10 \text{ cm}) \times \tan 40.6^\circ = 8.58 \text{ cm},$$

$$x = d_a \cot 30^\circ = 6.81 \cot 30^\circ = 11.8 \text{ cm}.$$ 

Hence,

$$x_2 = x - x_1 = 11.8 - 8.58 = 3.22 \text{ cm},$$

and 

$$d_2 = x_2 \cot 32.77^\circ = (3.22 \text{ cm}) \times \cot 32.77^\circ = 5 \text{ cm}.$$ 

Hence, $$d_t = (10 + 5) = 15 \text{ cm}.$$
Problem 8.28  Repeat Problem 8.27 for a wave in air with
\[ \overline{\mathbf{H}}^i = \hat{y} 2 \times 10^{-2} e^{-j(8x+6z)} \quad (\text{A/m}) \]
incident upon the planar boundary of a dielectric medium \((z \geq 0)\) with \(\varepsilon_r = 9.\)

Solution:
(a) \(\overline{\mathbf{H}}^i = \hat{y} 2 \times 10^{-2} e^{-j(8x+6z)}.\)
Since \(\mathbf{H}^i\) is along \(\hat{y}\), which is perpendicular to the plane of incidence, the wave is
TM polarized, or equivalently, its electric field vector is parallel polarized (parallel to
the plane of incidence).
(b) From Eq. (8.65b), the argument of the exponential is
\[-jk_1(x \sin \theta_i + z \cos \theta_i) = -j(8x + 6z).\]
Hence,
\[ k_1 \sin \theta_i = 8, \quad k_1 \cos \theta_i = 6, \]
from which we determine
\[ \theta_i = \tan^{-1} \left( \frac{8}{6} \right) = 53.13^\circ, \]
\[ k_1 = \sqrt{6^2 + 8^2} = 10 \quad \text{ (rad/m)}. \]
Also,
\[ \omega = u_p k = ck = 3 \times 10^8 \times 10 = 3 \times 10^9 \quad \text{ (rad/s)}. \]
(c)
\[ \eta_1 = \eta_0 = 377 \, \Omega, \]
\[ \eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_2}} \eta_0 = \frac{125.67}{3} = 41.89 \, \Omega, \]
\[ \theta_i = \sin^{-1} \left( \frac{\sin \theta_i}{\sqrt{\varepsilon_2}} \right) = \sin^{-1} \left( \frac{\sin 53.13^\circ}{\sqrt{9}} \right) = 53.13^\circ, \]
\[ \Gamma_i = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} = 0.30, \]
\[ \tau_i = (1 + \Gamma_i) \frac{\cos \theta_i}{\cos \theta_i} = 0.44. \]
In accordance with Eqs. (8.65a) to (8.65d), \(E_0^i = 2 \times 10^{-2} \eta_1\) and
\[ \overline{\mathbf{E}}^i = (\hat{x} \cos \theta_i - \hat{z} \sin \theta_i) 2 \times 10^{-2} \eta_1 e^{-j(8x+6z)} = (\hat{x} 4.52 - \hat{z} 6.03) e^{-j(8x+6z)}. \]
\(\tilde{E}'\) is similar to \(\tilde{E}'\) except for reversal of \(z\)-components and multiplication of amplitude by \(G_\parallel\). Hence, with \(G_\parallel = -0.30\),

\[
E' = \Re[\tilde{E}' e^{j\omega t}] = - (1.36 + 21.81) \cos(3 \times 10^9 t - 8x + 6z) \text{ V/m},
\]

\[
H' = \hat{y} 2 \times 10^{-2} \Gamma_\parallel \cos(3 \times 10^9 t - 8x + 6z)
- \hat{y} 0.6 \times 10^{-2} \cos(3 \times 10^9 t - 8x + 6z) \text{ A/m}.
\]

(d) In medium 2,

\[
k_2 = k_1 \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = 10\sqrt{\varepsilon} = 30 \text{ rad/m},
\]

\[
\theta_t = \sin^{-1}\left[\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \sin \theta_1\right] = \sin^{-1}\left[\frac{1}{3} \sin 53.13^\circ\right] = 15.47^\circ,
\]

and the exponent of \(E'\) and \(H'\) is

\[-jk_2(x \sin \theta_t + z \cos \theta_t) = -j30(x \sin 15.47^\circ + z \cos 15.47^\circ) = -j(8x + 28.91z).
\]

Hence,

\[
\tilde{E}' = (x \cos \theta_t - z \sin \theta_t) E_0^i \tau e^{-j(8x + 28.91z)}
= (0.96 - 0.27) 2 \times 10^{-2} \times 377 \times 0.44 e^{-j(8x + 28.91z)}
= (3.18 - 0.90) e^{-j(8x + 28.91z)},
\]

\[
\tilde{H}' = \hat{y} \frac{E_0^i \tau}{\eta_2} e^{-j(8x + 28.91z)}
= \hat{y} 2.64 \times 10^{-2} e^{-j(8x + 28.91z)},
\]

\[
E' = \Re\{\tilde{E}' e^{j\omega t}\}
= (3.18 - 0.90) \cos(3 \times 10^9 t - 8x - 28.91z) \text{ V/m},
\]

\[
H' = \hat{y} 2.64 \times 10^{-2} \cos(3 \times 10^9 t - 8x - 28.91z) \text{ A/m}.
\]

(e)

\[
S_{av} = \frac{|E_0^i|^2}{2\eta_2} = \frac{|H_0^i|^2}{2\eta_2} = \frac{(2.64 \times 10^{-2})^2}{2} \times 125.67 = 44 \text{ mW/m}^2.
\]
Problem 8.29  A plane wave in air with
\[ \vec{E}^i = (\hat{x} 9 - \hat{y} 4 - \hat{z} 6)e^{-j(2x + 3z)} \quad \text{(V/m)} \]
is incident upon the planar surface of a dielectric material, with \( \varepsilon_r = 2.25 \), occupying the half-space \( z \geq 0 \). Determine

(a) The incidence angle \( \theta_i \).

(b) The frequency of the wave.

(c) The field \( \vec{E}_r \) of the reflected wave.

(d) The field \( \vec{E}_t \) of the wave transmitted into the dielectric medium.

(e) The average power density carried by the wave into the dielectric medium.

Solution:

(a) From the exponential of the given expression, it is clear that the wave direction of travel is in the \( x-z \) plane. By comparison with the expressions in (8.48a) for perpendicular polarization or (8.65a) for parallel polarization, both of which have the same phase factor, we conclude that:

\[ k_1 \sin \theta_i = 2, \]
\[ k_1 \cos \theta_i = 3. \]
Hence,
\[ k_1 = \sqrt{2^2 + 3^2} = 3.6 \quad (\text{rad/m}) \]
\[ \theta_1 = \tan^{-1}(2/3) = 33.7^\circ. \]

Also,
\[ k_2 = k_1 \sqrt{\varepsilon_{tt2}} = 3.6 \sqrt{2.25} = 5.4 \quad (\text{rad/m}) \]
\[ \theta_2 = \sin^{-1} \left( \frac{1}{\sqrt{2.25}} \right) = 21.7^\circ. \]

(c) In order to determine the electric field of the reflected wave, we first have to determine the polarization of the wave. The vector argument in the given expression for \( \bar{E}^i \) indicates that the incident wave is a mixture of parallel and perpendicular polarization components. Perpendicular polarization has a \( \hat{y} \)-component only (see 8.46a), whereas parallel polarization has only \( \hat{x} \) and \( \hat{z} \) components (see 8.65a). Hence, we shall decompose the incident wave accordingly:
\[ \bar{E}^i = \bar{E}^i_\perp + \bar{E}^i_\parallel \]

with
\[ \bar{E}^i_\perp = -\hat{y} 4e^{-j(2x+3z)} \quad (\text{V/m}) \]
\[ \bar{E}^i_\parallel = (\hat{x} 9 - \hat{z} 6)e^{-j(2x+3z)} \quad (\text{V/m}) \]

From the above expressions, we deduce:
\[ E^i_{\perp 0} = -4 \text{ V/m} \]
\[ E^i_{\parallel 0} = \sqrt{9^2 + 6^2} = 10.82 \text{ V/m}. \]

Next, we calculate \( \Gamma \) and \( \tau \) for each of the two polarizations:
\[ \Gamma_\perp = \frac{\cos \theta_1 - \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2 \theta_1}}{\cos \theta_1 + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2 \theta_1}} \]
Using $\theta_i = 33.7^\circ$ and $\varepsilon_2/\varepsilon_1 = 2.25/1 = 2.25$ leads to:

$$\Gamma_\bot = -0.25$$
$$\tau_\bot = 1 + \Gamma_\bot = 0.75.$$  

Similarly,

$$\Gamma_\bot = -\left(\frac{\varepsilon_2/\varepsilon_1}{\varepsilon_2/\varepsilon_1} \cos \theta_i + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2 \theta_i}\right) - \left(\frac{\varepsilon_2/\varepsilon_1}{\varepsilon_2/\varepsilon_1} \cos \theta_i + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2 \theta_i}\right) = -0.15,$$

$$\tau_\parallel = (1 + \Gamma_\parallel) \cos \frac{\theta_i}{\cos \theta_i} = (1 - 0.15) \cos 33.7^\circ \cos 21.7^\circ = 0.76.$$  

The electric fields of the reflected and transmitted waves for the two polarizations are given by (8.49a), (8.49c), (8.65c), and (8.65e):

$$\tilde{E}_\perp^r = \hat{y} E_{\perp,0} e^{-jk_1(x \sin \theta_t - z \cos \theta_t)}$$
$$\tilde{E}_\perp^t = \hat{y} E_{\perp,0} e^{-jk_2(x \sin \theta_t + z \cos \theta_t)}$$
$$\tilde{E}_\parallel^r = (\hat{x} \cos \theta_t + \hat{z} \sin \theta_t) E_{\parallel,0} e^{-jk_1(x \sin \theta_t - z \cos \theta_t)}$$
$$\tilde{E}_\parallel^t = (\hat{x} \cos \theta_t - \hat{z} \sin \theta_t) E_{\parallel,0} e^{-jk_2(x \sin \theta_t + z \cos \theta_t)}$$

Based on our earlier calculations:

$$\theta_r = \theta_i = 33.7^\circ$$
$$\theta_t = 21.7^\circ$$
$$k_1 = 3.6 \text{ rad/m, } \quad k_2 = 5.4 \text{ rad/m,}$$
$$E_{\parallel,0} = \Gamma_\bot E_{\perp,0} = (-0.25) \times (-4) = 1 \text{ V/m.}$$
$$E_{\perp,0} = \tau_\bot E_{\perp,0} = 0.75 \times (-4) = -3 \text{ V/m.}$$
$$E_{\parallel,0} = \Gamma_\parallel E_{\parallel,0} = (-0.15) \times 10.82 = -1.62 \text{ V/m.}$$
$$E_{\parallel,0} = \tau_\parallel E_{\parallel,0} = 0.76 \times 10.82 = 8.22 \text{ V/m.}$$

Using the above values, we have:

$$\tilde{E}^r = \tilde{E}_\perp^r + \tilde{E}_\parallel^r$$
$$= (\hat{x} E_{\parallel,0} \cos \theta_r + \hat{y} E_{\perp,0} + \hat{z} E_{\parallel,0} \sin \theta_r) e^{-j(2x - 3z)}$$
$$= (-\hat{x} 1.35 + \hat{y} - 0.90) e^{-j(2x - 3z)} \quad (\text{V/m}).$$
(d) \[
\vec{E}^t = \vec{E}_\perp^t + \vec{E}_\parallel^t \\
= (\hat{x}7.65 - \hat{y}3 - \hat{z}23.05)e^{-j(2\pi + 5\pi)} \text{ (V/m)}.
\]

(e) \[
S^t = \frac{|E_0^t|^2}{2\eta_2} \\
|E_0^t|^2 = (7.65)^2 + 3^2 + (3.05)^2 = 76.83 \\
\eta_2 = \frac{\eta_0}{\sqrt{\epsilon_{t2}}} = \frac{377}{1.5} = 251.3 \Omega \\
S^t = \frac{76.83}{2 \times 251.3} = 152.86 \text{ (mW/m}^2).
Problem 9.1  A center-fed Hertzian dipole is excited by a current \( I_0 = 20 \text{ A} \). If the dipole is \( \lambda / 50 \) in length, determine the maximum radiated power density at a distance of 1 km.

Solution: From Eq. (9.14), the maximum power density radiated by a Hertzian dipole is given by

\[
S_0 = \frac{n_0 k^2 I_0^2 l^2}{32 \pi^2 R^2} = \frac{377 \times \left(2\pi/\lambda\right)^2 \times 20^2 \times (\lambda/50)^2}{32 \pi^2 (10^3)^2} = 7.6 \times 10^{-6} \text{ W/m}^2 = 7.6 \text{ (\mu W/m}^2). \]
**Problem 9.14** For a short dipole with length \( l \) such that \( l \ll \lambda \), instead of treating the current \( \tilde{I}(z) \) as constant along the dipole, as was done in Section 9-1, a more realistic approximation that ensures the current goes to zero at the dipole ends is to describe \( \tilde{I}(z) \) by the triangular function

\[
\tilde{I}(z) = \begin{cases} 
I_0(1 - 2z/l), & \text{for } 0 \leq z \leq l/2 \\
I_0(1 + 2z/l), & \text{for } -l/2 \leq z \leq 0
\end{cases}
\]

as shown in Fig. P9.14. Use this current distribution to determine the following:

(a) The far-field \( \tilde{E}(R, \theta, \phi) \).

(b) The power density \( S(R, \theta, \phi) \).

(c) The directivity \( D \).

(d) The radiation resistance \( R_{\text{rad}} \).

**Solution:**

(a) When the current along the dipole was assumed to be constant and equal to \( I_0 \), the vector potential was given by Eq. (9.3) as:

\[
\tilde{A}(R) = \hat{z} \frac{\mu_0}{4\pi} \left( \frac{e^{-jkR}}{R} \right) \int_{-l/2}^{l/2} I_0 dz.
\]

If the triangular current function is assumed instead, then \( I_0 \) in the above expression should be replaced with the given expression. Hence,

\[
\tilde{A} = \hat{z} \frac{\mu_0}{4\pi} \left( \frac{e^{-jkR}}{R} \right) \left[ \int_{0}^{l/2} \left( 1 - \frac{2z}{l} \right) dz + \int_{-l/2}^{0} \left( 1 + \frac{2z}{l} \right) dz \right] = \hat{z} \frac{\mu_0 I_0 l}{8\pi} \left( \frac{e^{-jkR}}{R} \right),
\]

which is half that obtained for the constant-current case given by Eq. (9.3). Hence, the expression given by (9.9a) need only be modified by the factor of 1/2:

\[
\tilde{E} = \hat{\theta} \tilde{E}_\theta = \hat{\theta} \frac{j \mu_0 k \eta_0}{8\pi} \left( \frac{e^{-jkR}}{R} \right) \sin \theta.
\]
(b) The corresponding power density is

\[ S(R, \theta) = \frac{|\mathbf{E}_\theta|^2}{2\eta_0} = \left( \frac{\eta_0 k^2 I_0^2 l^2}{128\pi^2 R^2} \right) \sin^2 \theta. \]

(c) The power density is 4 times smaller than that for the constant current case, but the reduction is true for all directions. Hence, \( D \) remains unchanged at 1.5.

(d) Since \( S(R, \theta) \) is 4 times smaller, the total radiated power \( P_{\text{rad}} \) is 4-times smaller. Consequently, \( R_{\text{rad}} = 2P_{\text{rad}}/I_0^2 \) is 4 times smaller than the expression given by Eq. (9.35); that is,

\[ R_{\text{rad}} = 20\pi^2 \left( \frac{l}{\lambda} \right)^2 \quad (\Omega). \]
Problem 9.18  A car antenna is a vertical monopole over a conducting surface. Repeat Problem 9.5 for a 1-m–long car antenna operating at 1 MHz. The antenna wire is made of aluminum with $\mu_\text{c} = \mu_0$ and $\sigma_\text{c} = 3.5 \times 10^7 \text{ S/m}$, and its diameter is 1 cm.

Solution:

(a) Following Example 9-3, $\lambda = c/f = (3 \times 10^8 \text{ m/s})/(10^6 \text{ Hz}) = 300 \text{ m}$. As $l/\lambda = 2 \times (1 \text{ m})/(300 \text{ m}) = 0.0067$, this antenna is a short (Hertzian) monopole. From Section 9-3.3, the radiation resistance of a monopole is half that for a corresponding dipole. Thus,

$$R_\text{rad} = \frac{1}{2} 80\pi^2 \left( \frac{l}{\lambda} \right)^2 = 40\pi^2 (0.0067)^2 = 17.7 \text{ (m}\Omega) ,$$

$$R_\text{loss} = \frac{l}{2\pi a} \sqrt{\frac{\pi f \mu_\text{c}}{\sigma_\text{c}}} = \frac{1 \text{ m}}{\pi (10^{-2} \text{ m})} \sqrt{\frac{\pi (10^6 \text{ Hz})(4\pi \times 10^{-7} \text{ H/m})}{3.5 \times 10^7 \text{ S/m}}} = 10.7 \text{ m}\Omega ,$$

$$\xi = \frac{R_\text{rad}}{R_\text{rad} + R_\text{loss}} = \frac{17.7 \text{ m}\Omega}{17.7 \text{ m}\Omega + 10.7 \text{ m}\Omega} = 62\% .$$

(b) From Example 9-2, a Hertzian dipole has a directivity of 1.5. The gain, from Eq. (9.29), is $G = \xi D = 0.62 \times 1.5 = 0.93 = -0.3 \text{ dB}$.

(c) From Eq. (9.30a),

$$I_0 = \sqrt{\frac{2P_\text{rad}}{R_\text{rad}}} = \sqrt{\frac{2(80 \text{ W})}{17.7 \text{ m}\Omega}} = 95 \text{ A} ,$$

and from Eq. (9.31),

$$P_\text{t} = \frac{P_\text{rad}}{\xi} = \frac{80 \text{ W}}{0.62} = 129.2 \text{ W} .$$
Problem 9.21  A half-wave dipole TV broadcast antenna transmits 1 kW at 50 MHz. What is the power received by a home television antenna with 3-dB gain if located at a distance of 30 km?

Solution: At $f = 50$ MHz, $\lambda = c/f = (3 \times 10^8 \text{ m/s})/(50 \times 10^6 \text{ Hz}) = 6 \text{ m}$, for which a half wave dipole, or larger antenna, is very reasonable to construct. Assuming the TV transmitter to have a vertical half wave dipole, its gain in the direction of the home would be $G_t = 1.64$. The home antenna has a gain of $G_r = 3 \text{ dB} = 2$. From the Friis transmission formula (Eq. (9.75)):

$$P_{\text{rec}} = P_t \frac{\lambda^2 G_t G_r}{(4\pi)^2 R^2} = 10^3 \frac{(6 \text{ m})^2 \times 1.64 \times 2}{(4\pi)^2(30 \times 10^3 \text{ m})^2} = 8.3 \times 10^{-7} \text{ W}.$$
Problem 9.22  A 150-MHz communication link consists of two vertical half-wave dipole antennas separated by 2 km. The antennas are lossless, the signal occupies a bandwidth of 3 MHz, the system noise temperature of the receiver is 600 K, and the desired signal-to-noise ratio is 17 dB. What transmitter power is required?

Solution: From Eq. (9.77), the receiver noise power is

\[ P_n = KT_{sys}B = 1.38 \times 10^{-23} \times 600 \times 3 \times 10^6 = 2.48 \times 10^{-14} \text{ W}. \]

For a signal to noise ratio \( S_n = 17 \text{ dB} = 50 \), the received power must be at least

\[ P_{rec} = S_n P_n = 50(2.48 \times 10^{-14} \text{ W}) = 1.24 \times 10^{-12} \text{ W}. \]

Since the two antennas are half-wave dipoles, Eq. (9.47) states \( D_t = D_r = 1.64 \), and since the antennas are both lossless, \( G_t = D_t \) and \( G_r = D_r \). Since the operating frequency is \( f = 150 \text{ MHz} \), \( \lambda = c/f = (3 \times 10^8 \text{ m/s})/(150 \times 10^6 \text{ Hz}) = 2 \text{ m} \). Solving the Friis transmission formula (Eq. (9.75)) for the transmitted power:

\[ P_t = P_{rec} \frac{(4\pi)^2 R^2}{\lambda^2 G_t G_r} = 1.24 \times 10^{-12} \frac{(4\pi)^2 (2 \times 10^3 \text{ m})^2}{(2 \text{ m})^2 (1.64)(1.64)} = 75 \text{ (\mu W)}. \]
Problem 9.23  Consider the communication system shown in Fig. P9.23, with all components properly matched. If $P_t = 10$ W and $f = 6$ GHz:

(a) What is the power density at the receiving antenna (assuming proper alignment of antennas)?

(b) What is the received power?

(c) If $T_{sys} = 1,000$ K and the receiver bandwidth is 20 MHz, what is the signal-to-noise ratio in decibels?

![Figure P9.23: Communication system of Problem 9.23.](image)

Solution:

(a) $G_t = 20$ dB = 100, $G_r = 23$ dB = 200, and $\lambda = c/f = 5$ cm. From Eq. (9.72),

$$S_r = G_t \frac{P_t}{4\pi R^2} = \frac{10^2 \times 10}{4\pi \times (2 \times 10^4)^2} = 2 \times 10^{-7} \text{ (W/m}^2\text{).}$$

(b)

$$P_{rec} = P_t G_t G_r \left( \frac{\lambda}{4\pi R} \right)^2 = 10 \times 100 \times 200 \times \left( \frac{5 \times 10^{-2}}{4\pi \times 2 \times 10^4} \right)^2 = 7.92 \times 10^{-9} \text{ W.}$$

(c)

$$P_n = KT_{sys}B = 1.38 \times 10^{-23} \times 10^3 \times 2 \times 10^7 = 2.76 \times 10^{-13} \text{ W},$$

$$S_n = \frac{P_{rec}}{P_n} = \frac{7.92 \times 10^{-9}}{2.76 \times 10^{-13}} = 2.87 \times 10^4 = 44.6 \text{ dB.}$$