2D Plane Wave:

The direction of propagation is given by the wave vector

\[ \mathbf{\hat{r}} = k \cos \theta \hat{z} + k \sin \theta \hat{x} = k\hat{z} + k\hat{x} \]

where

\[ |\mathbf{\hat{r}}| = k = \sqrt{k_z^2 + k_x^2} \]

Given the observation point \( \mathbf{\hat{r}} = \hat{x} + \hat{z} \)

the function

\[ e^{-j \mathbf{\hat{r}} \cdot \mathbf{\hat{r}}'} = e^{-j(k_x x + k_z z)} = e^{-j(k \sin \theta_x x + k \cos \theta_z z)} \]
\[ = e^{-j(k \sin \theta_x \mathbf{\hat{r}} \cdot \mathbf{\hat{r}}' + k \cos \theta_z \mathbf{\hat{r}} \cdot \mathbf{\hat{r}}')} \]
\[ = e^{-jk r (\cos \theta \cos \theta_z + \sin \theta \sin \theta_z)} \]
\[ = e^{-jk r \cos (\theta - \theta_z)} \]

has constant values along the line defined by \( \mathbf{\hat{r}} \cdot \mathbf{\hat{r}}' = \text{constant} \), which is a line orthogonal to the direction \( \mathbf{\hat{r}} \)

\[ \mathbf{\hat{r}} \cdot \mathbf{\hat{r}}' = k r \cos (\theta - \theta_z) \]

The 2D plane wave solution has two possible forms:
Parallel Polarization: $\vec{E}$ lies in $xz$ plane. $\vec{H}$ perpendicular, along $\hat{y}$

$$\vec{H}_\omega(x, y, z) = \frac{E_0}{\eta} e^{-j\left(k_x x + k_z z\right)} \hat{y}$$

$$\vec{E}_\omega(x, y, z) = \frac{\nabla \times \vec{H}_\omega}{j \omega \varepsilon} = - \frac{\partial_z H_y \hat{x}}{j \omega \varepsilon} + \frac{\partial_x H_y \hat{z}}{j \omega \varepsilon}$$

$$\partial_z H_y = -j k_z \frac{E_0}{\eta} e^{-j\left(k_x x + k_z z\right)}$$

$$\partial_x H_y = -j k_x \frac{E_0}{\eta} e^{-j\left(k_x x + k_z z\right)}$$

Thus

$$\vec{E}_\omega(x, y, z) = \left( \frac{k_z^2}{\omega \varepsilon} \hat{x} - \frac{k_x}{\omega \varepsilon} \hat{z} \right) \frac{E_0}{\eta} e^{-j\left(k_x x + k_z z\right)}$$

$$= (\cos \theta_i \hat{x} - \sin \theta_i \hat{z}) \frac{k}{\omega \mu} \frac{E_0}{\eta} e^{-j\left(k_x x + k_z z\right)}$$

To determine $k/\omega \varepsilon$, we also calculate

$$\vec{H}_\omega = \frac{\nabla \times \vec{E}_\omega}{-j \omega \mu} = \left( \frac{\partial_z E_z - \partial_x E_x}{-j \omega \mu} \right) \hat{y}$$

$$= \left( -j \frac{k_z^2}{\omega \varepsilon} - j \frac{k_x^2}{\omega \varepsilon} \right) \frac{E_0}{\eta} e^{-j\left(k_x x + k_z z\right)} \hat{y}$$

$$= \left( \frac{k_z^2 + k_x^2}{\omega^2 \varepsilon \mu} \right) \frac{E_0}{\eta} e^{-j\left(k_x x + k_z z\right)} \hat{y}$$

Which means

$$k^2 = k_z^2 + k_x^2 = \omega^2 \varepsilon \mu$$
Therefore we can write for \underline{parallel pol}:

\[
\vec{E}_\omega(x, y, z) = (\cos \theta_c \hat{x} - \sin \theta_c \hat{z}) \vec{E}_0 \ e^{-j(k_x x + k_z z)}
\]

\[
\vec{H}_\omega(x, y, z) = \frac{\vec{E}_0}{\eta} \ e^{-j(k_x x + k_z z)} \ y
\]

\[
k_x = k \sin \theta_c \quad k_z = k \cos \theta_c
\]

\[
k^2 = k_x^2 + k_z^2 = \omega^2 \epsilon \mu
\]

\[
\eta = \frac{k}{\omega \epsilon} = \sqrt{\frac{\mu}{\epsilon}} \quad = \frac{\omega \mu}{k}
\]

\underline{Perpendicular Polarization:} \quad \vec{E} \ \text{lies along} \ y \quad \vec{H} \ \text{lies in} \ xz \ \text{plane}

\[
\vec{E}_\omega(x, y, z) = \vec{E}_0 \ e^{-j(k_x x + k_z z)} \ y
\]

\[
\vec{H}_\omega(x, y, z) = \nabla \times \vec{E}_\omega = \frac{\partial E_y}{\partial z} \hat{z} - \frac{\partial E_z}{\partial x} \hat{x}
\]

\[
= \left( \frac{k_x}{\omega \mu} \hat{z} - \frac{k_z}{\omega \mu} \hat{x} \right) \vec{E}_0 \ e^{-j(k_x x + k_z z)}
\]

\[
= (\sin \theta_c \hat{z} - \cos \theta_c \hat{x}) \vec{E}_0 \ e^{-j(k_x x + k_z z)}
\]

Note: If \( \theta_c = 0 \), then both cases reduce

to 1D plane waves we have studied

Note: \( \nabla \cdot \vec{D} = 0 \), \( D \cdot \vec{B} = 0 \) for both waves, e.g.

for parallel pol:

\[
\nabla \cdot \vec{D} = \epsilon (\partial_x E_x + \partial_z E_z) = \epsilon \left[ (-j k_x) \frac{k_z}{\omega \epsilon} + (-j k_z) \frac{k_x}{\omega \epsilon} \right] \vec{E}_0 e^{-j(k_x x + k_z z)}
\]
\[ D \cdot B = \mu \frac{\partial^2}{\partial y^2} H_y = 0. \]

**Power flow in parallel pol.**

\[ \langle S_{\omega} \rangle = \frac{1}{2} \text{Re} \left\{ \frac{E_{\omega} \times H_{\omega}^*}{\omega} \right\} \]

\[ = \frac{1}{2} \text{Re} \left\{ (k_x \hat{x} - k_z \hat{z}) \frac{H_0}{\omega e^{j(k_x x + k_z z)}} \times \frac{H_0^*}{\omega e^{j(k_x x + k_z z)}} \right\} \]

\[ = \frac{1}{2} \left| \frac{H_0}{\omega e^{j(k_x x + k_z z)}} \right|^2 \left[ k_z (\hat{x} \times \hat{y}) - k_x (\hat{z} \times \hat{y}) \right] \]

\[ = \frac{1}{2} \left| \frac{H_0}{\omega e^{j(k_x x + k_z z)}} \right|^2 \left( k_z \hat{z} + k_x \hat{x} \right) \]

\[ = \frac{1}{2} \left| \frac{H_0}{\omega e^{j(k_x x + k_z z)}} \right|^2 \hat{k} = \frac{1}{2} \eta \left| H_0 \right|^2 \hat{k} \]

**Perpendicular pol.**

\[ \langle S_{\omega} \rangle = \frac{1}{2} \text{Re} \left\{ \frac{E_{\omega} \times H_{\omega}^*}{\omega} \right\} \]

\[ = \frac{1}{2} \text{Re} \left\{ \frac{E_0}{\omega e^{j(k_x x + k_z z)}} \hat{y} \times \frac{E_0^*}{\omega e^{j(k_x x + k_z z)}} \hat{y} \times \frac{E_0}{\omega e^{j(k_x x + k_z z)}} \hat{k} \right\} \]

\[ = \frac{1}{2} \left| \frac{E_0}{\omega e^{j(k_x x + k_z z)}} \right|^2 \left[ k_x (\hat{y} \times \hat{z}) - k_z (\hat{y} \times \hat{x}) \right] \]

\[ = \frac{1}{2} \left| \frac{E_0}{\omega e^{j(k_x x + k_z z)}} \right|^2 \left( k_z \hat{z} + k_x \hat{x} \right) \]

\[ = \frac{1}{2} \left| \frac{E_0}{\omega e^{j(k_x x + k_z z)}} \right|^2 \hat{k} = \frac{1}{2} \frac{\left| E_0 \right|^2}{\eta} \hat{k} \]

Power flow is in the d.o.p.
Oblique Scattering from Interface:

Plane of Incidence is defined by D.O.P.

of incident wave and normal to
the interface

\[ E_{i1}, \mu_1 \rightarrow \hat{n}, E_{z1}, \mu_2 \rightarrow \hat{z} \]

Parallel Pol

Perpendicular Pol

Have to write down reflected and transmitted waves and apply the EM Boundary Conditions

\[ \begin{align*}
E_{\tan_{1,2}} &= E_{\tan_{1,2}} \\
H_{\tan_{1,2}} &= H_{\tan_{1,2}}
\end{align*} \]

Parallel Pol:

\[ \begin{align*}
\vec{E}_r &= \hat{k}_r \\
\vec{H}_r &= \hat{\rho}_r \\
\vec{E}_t &= \hat{k}_t \\
\vec{H}_t &= \hat{\rho}_t
\end{align*} \]

\[ \begin{align*}
\vec{R}_i &= k_1 (\cos \Theta_i \hat{z} + \sin \Theta_i \hat{x}) \\
\vec{R}_r &= k_1 (-\cos \Theta_r \hat{z} + \sin \Theta_r \hat{x}) \\
\vec{R}_t &= k_2 (\cos \Theta_t \hat{z} + \sin \Theta_t \hat{x})
\end{align*} \]

Angles defined with respect to normal.
Incident
\[ H_{\text{inc}} = \hat{y} \frac{E_0}{\eta_1} e^{-jk_1(c\cos \theta_z + s \mu \theta_z x)} \]
\[ E_{\text{inc}} = (c\cos \theta_z \hat{x} - s \mu \theta_z \hat{z}) \frac{E_0}{\eta_1} e^{-jk_1(c\cos \theta_z + s \mu \theta_z x)} \]

Reflected
\[ H_{\text{ref}} = -\hat{y} \frac{E_0}{\eta_1} e^{-jk_1(-c\cos \theta_r + s \mu \theta_r x)} \]
\[ E_{\text{ref}} = (c\cos \theta_r \hat{x} + s \mu \theta_r \hat{z}) \frac{E_0}{\eta_1} e^{-jk_1(-c\cos \theta_r + s \mu \theta_r x)} \]
where \( T_{\parallel} \) is the parallel polar reflectance coefficient.

Transmitted
\[ H_{\text{trans}} = \hat{y} \frac{T_{\parallel} E_0}{\eta_2} e^{-jk_2(c\cos \theta_\perp \hat{z} + s \mu \theta_\perp \hat{x})} \]
\[ E_{\text{trans}} = (c\cos \theta_\perp \hat{x} - s \mu \theta_\perp \hat{z}) \frac{T_{\parallel} E_0}{\eta_2} e^{-jk_2(c\cos \theta_\perp + s \mu \theta_\perp x)} \]

Boundary conditions: Applied at \( z = 0 \)
tangential
\[ [\vec{E}_o \text{inc} (z=0) + \vec{E}_o \text{ref} (z=0)] = [\vec{E}_o \text{trans} (z=0)] \]
tangential
\[ [\vec{H}_o \text{inc} (z=0) + \vec{H}_o \text{ref} (z=0)] \times = [\vec{H}_o \text{trans} (z=0)] \times \]

Gives
\[ \frac{\cos \theta_z E_0 e^{-jk_1 s \mu \theta_z x} + \cos \theta_r T_{\parallel} E_0 e^{-jk_1 s \mu \theta_r x}}{\eta_1} \]
\[ = \cos \theta_\perp T_{\parallel} E_0 e^{-jk_2 s \mu \theta_\perp x} \]
\[ \frac{E_0 e^{-jk_1 s \mu \theta_z x} - T_{\parallel} E_0 e^{-jk_1 s \mu \theta_r x}}{\eta_1} = \frac{T_{\parallel} E_0 e^{-jk_2 s \mu \theta_\perp x}}{\eta_2} \]

For these relations to be true for all \( x \):
\[ k_1 s \mu \theta_z = k_1 s \mu \theta_r = k_2 s \mu \theta_\perp \]

This gives
\[ \theta_i = \theta_r \]  \quad \text{Law of Reflection}

\[ k_1 \sin \theta_i = k_2 \sin \theta_r \]  \quad \text{SNELL'S LAW}

Then

\[ \cos \theta_i \left(1 + \frac{n_1}{n_2}\right) = \cos \theta_r \frac{T_{11}}{T_{11}} \]

\[ 1 - \frac{n_1}{n_2} = \frac{n_1}{n_2} \frac{T_{11}}{T_{11}} \]

or

\[ 2 = \left(\frac{\cos \theta_r}{\cos \theta_i} + \frac{n_1}{n_2}\right) \frac{T_{11}}{T_{11}} \]

\[ T_{11} = \frac{2 \frac{n_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i}} \]

and

\[ \frac{n_1}{n_2} \frac{T_{11}}{T_{11}} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} \]

**Note:**

\[ \langle \vec{S}_{\omega}^{\text{inc.}} \rangle = \frac{1 |E_0|^2}{2 \eta_1} \hat{k}_i \]

\[ \langle \vec{S}_{\omega}^{\text{refl.}} \rangle = |\Gamma_{11}|^2 |E_0|^2 \hat{k}_r \]

\[ \langle \vec{S}_{\omega}^{\text{trans.}} \rangle = |T_{11}|^2 |E_0|^2 \hat{k}_t \]

Then

\[ \pm \cdot \langle \vec{S}_{\omega}^{\text{inc.}} + \vec{S}_{\omega}^{\text{refl.}} \rangle = \pm \cdot \langle \vec{S}_{\omega}^{\text{trans.}} \rangle \]

\[ \pm \cdot \langle \vec{S}_{\omega}^{\text{inc.}} + \vec{S}_{\omega}^{\text{refl.}} \rangle = \pm \cdot \langle \vec{S}_{\omega}^{\text{trans.}} \rangle \]

Power-flow components must be conserved.
\[
\frac{|E_0|^2}{2\gamma_1} \cos \theta_i - \frac{|T_{in}|^2|E_o|^2}{2\gamma_1} \cos \theta_r = \frac{|T_{in}|^2|E_o|^2}{2\gamma_2} \cos \theta_t \]

\[
\frac{|E_0|^2}{2\gamma_1} \left[ 1 - |T_{in}|^2 \right] = \frac{\cos \theta_t}{\cos \theta_i} \frac{|T_{in}|^2}{2\gamma_2} \frac{|E_0|^2}{2\gamma_2}
\]

or
\[
P_{inc} \left[ 1 - |T_{in}|^2 \right] = P_{trans}
\]

The \( x \) components of the power flow simply provide a relationship between the angle values and the reflection/transmission coefficients, i.e.

\[
\frac{|E_0|^2}{2\gamma_1} \sin \theta_i + \frac{|T_{in}|^2}{2\gamma_1} \frac{|E_0|^2}{2\gamma_2} \sin \theta_r = \frac{|T_{in}|^2}{2\gamma_2} \frac{|E_0|^2}{2\gamma_2} \sin \theta_t
\]

gives
\[
(1 + |T_{in}|^2) \sin \theta_i = \frac{\gamma_1}{\gamma_2} |T_{in}|^2 \sin \theta_t
\]

which combined with the \( x \) projection:
\[
(1 - |T_{in}|^2) \cos \theta_i = \frac{\gamma_1}{\gamma_2} |T_{in}|^2 \cos \theta_t
\]

yields
\[
\frac{1 - |T_{in}|^2}{1 + |T_{in}|^2} = \frac{\cos \theta_t}{\cos \theta_i} \frac{\sin \theta_i}{\sin \theta_t} = \frac{\tan \theta_i}{\tan \theta_t}
\]

or
\[
|T_{in}|^2 = \frac{\tan \theta_t - \tan \theta_i}{\tan \theta_t + \tan \theta_i}
\]

Clearly, lots of relations can be obtained between the trig functions of the angles and \( T_{in}, T_{in} \).
Reperpendicular Polarization:

All same k's, angles, impedences. Different picture:

\[
\begin{align*}
\vec{E}_{inc} & = \hat{\imath} E_0 e^{-j k_1 (\cos \theta_i z + \sin \theta_i x)} \\
\vec{H}_{inc} & = (\sin \theta_i \hat{\jmath} - \cos \theta_i \hat{x}) \frac{E_0}{\eta_1} e^{-j k_1 (\cos \theta_i z + \sin \theta_i x)} \\
\vec{E}_{ref} & = \hat{\imath} \Gamma E_0 e^{-j k_1 (-\cos \theta_r z + \sin \theta_r x)} \\
\vec{H}_{ref} & = (\sin \theta_r \hat{\jmath} + \cos \theta_r \hat{x}) \frac{\Gamma E_0}{\eta_1} e^{-j k_1 (-\cos \theta_r z + \sin \theta_r x)} \\
\vec{E}_{trans} & = \hat{\imath} \Gamma \parallel E_0 e^{-j k_2 (\cos \theta_t z + \sin \theta_t x)} \\
\vec{H}_{trans} & = (\sin \theta_t \hat{\jmath} - \cos \theta_t \hat{x}) \frac{\parallel E_0}{\eta_2} e^{-j k_2 (\cos \theta_t z + \sin \theta_t x)}
\end{align*}
\]

Boundary Conditions Require:

\[
\begin{align*}
E_0 e^{-j k_1 \sin \theta_i x} + \Gamma E_0 e^{-j k_1 \sin \theta_r x} & = \Gamma \parallel E_0 e^{-j k_2 \sin \theta_t x} \\
-\cos \theta_i \frac{E_0}{\eta_1} e^{-j k_1 \sin \theta_i x} + \cos \theta_r \frac{\parallel E_0}{\eta_1} e^{-j k_1 \sin \theta_r x} & = -\cos \theta_t \frac{\parallel E_0}{\eta_2} e^{-j k_2 \sin \theta_t x}
\end{align*}
\]
obtain law of reflection, Snell's Law again:

\[ \theta_r = \theta_i, \quad k_1 \sin \theta_i = k_2 \sin \theta_t \]

and

\[ 1 + \Gamma_\perp = \frac{T_\perp}{T_\perp} \]

\[ 1 - \Gamma_\perp = \frac{\eta_1}{\eta_2} \frac{\cos \theta_t}{\cos \theta_i} \]

Thus

\[ T_\perp = \frac{2 \eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} \]

\[ \Gamma_\perp = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} \]

Note: The directed power flow again gives

\[ \text{pinc} \left[ 1 - 1|\Gamma_\perp|^2 \right] = P_{\text{trans}} \]

Note: From the power flow arguments we have

\[ |\Gamma_{\text{II}}|^2 + \frac{\eta_1}{\eta_2} \frac{\cos \theta_t}{\cos \theta_i} |\Gamma_{\text{II}}|^2 = 1 \]

\[ |\Gamma_{\perp}|^2 + \frac{\eta_1}{\eta_2} \frac{\cos \theta_t}{\cos \theta_i} |\Gamma_{\perp}|^2 = 1 \]

or

\[ R + T = 1 \]

\[
\begin{array}{c}
\text{power reflection coefficient} \\
\text{power transmitted coefficient} \\
\text{normalized incident power}
\end{array}
\]