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A new procedure for specifying nonradiating current distributions and the fields they produce

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This paper reports a new procedure for specifying monochromatic nonradiating (NR) current distributions (NR sources) and the electric and magnetic fields they produce (NR fields). Vector spherical harmonics and a Fourier–Bessel series are used to derive a new vector spherical-wave expansion for continuous NR fields confined within a spherical volume. The analysis yields complete orthogonal sets in terms of which all such NR fields can be expanded. By making use of a Maxwell operator representation for NR current distributions, we obtain a new series expansion for NR current distributions confined within a spherical volume. The analysis also yields complete sets for such NR current distributions. The developed theory is illustrated with special cases. © *2000 American Institute of Physics.*
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I. INTRODUCTION

Classical current distributions which do not radiate [nonradiating (NR) sources] have been studied since the early days of electromagnetic theory (see Ref. 1 for a review and relevant references). Interest in such a class of sources originated from their connection with certain aspects of classical electron theory, primarily the question of the electromagnetic self-force and radiation reaction.[2] NR sources were used in interesting papers by Schott[3,4] and Bohm and Weinstein[5] and, more recently, by Goedecke,[6] in efforts to model charged particles and atoms as manifestations of NR source states. In more recent years, most of the (renewed) interest in NR sources has been linked to their role in inverse source and inverse scattering theories where they arise as the null space of the mapping from the source (scatterer) to the field.[7–9] Investigations on this subject have addressed both scalar[10–14] and electromagnetic sources,[1,15–17] including both deterministic and random sources.[18–20] The vast majority of workers have focused on the scalar formulation, as opposed to the vector, electromagnetic formulation. The latter is the focus of our presentation.

This paper reports a new procedure for specifying monochromatic NR current distributions and the electric and magnetic fields they produce (NR fields). Our analysis is based on standard vector spherical harmonics and a Fourier–Bessel series and yields new representations and basis functions for NR sources and fields confined within a spherical volume. The results derived in this paper provide a systematic way to construct such wave objects and are therefore relevant to computational aspects of inverse source/inverse scattering reconstruction. In fact, part of the motivation for the research reported here was provided by the need for representational tools for NR source components of scattering objects in certain source-type integral equation (STIE) methods.[21–23] In Ref. 23, basis functions to represent NR sources in rectangular coordinates were derived, and applied to the problem of reconstructing, via inverse scattering surveys, the constitutive properties of an unknown object. The spherical coordinate counterparts of the NR source results in Ref. 23 were developed first for scalar, spherically symmetric sources in Ref. 13, and

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extended later to the nonspherically symmetric case in a recent contribution coauthored by one of us (E.A.M.).[24] The present work generalizes, to the electromagnetic case, that in Ref. 24.

The remainder of this paper is organized as follows. In Sec. II, localized NR current distributions and the fields they produce are characterized as solutions of an overspecified boundary value problem of the inhomogeneous vector wave equation. This characterization is based on a well-known Maxwell operator representation for NR current distributions derived first in Ref. 15. In Secs. III and IV, a new method is developed for specifying NR current distributions and NR fields confined within a spherical volume. In both sections we impose certain continuity and differentiability restrictions on NR sources and fields which can, however, be relaxed by dealing with the various vector differential operators in a weak derivative or distributional sense. In Sec. III, vector spherical harmonics and a Fourier–Bessel series are used to derive a new series representation for continuous NR fields confined within a spherical volume. The analysis also yields complete orthogonal sets in terms of which all such NR fields can be expanded. In Sec. IV, we derive a new series representation for the NR current distributions associated with the NR fields in Sec. III. Our analysis also yields complete sets for all such NR current distributions. In Sec. V, the general theory is applied to the special cases of spherically symmetric NR sources and NR sources with dipolar angular dependence (NR loops of current contained within a spherical region). Section VI contains our concluding remarks.

II. THE DEVANEY–WOLF REPRESENTATION

In Gaussian system of units, the Maxwell equations in free space reduce, under time-harmonic conditions, to[25]

$$\begin{aligned}
\nabla\cdot\mathbf{E}(\mathbf{r}) &= 4\pi\rho(\mathbf{r}),\\
\nabla\cdot\mathbf{H}(\mathbf{r}) &= 0,\\
\nabla\times\mathbf{E}(\mathbf{r}) &= i\frac{\omega}{c}\mathbf{H}(\mathbf{r}),\\
\nabla\times\mathbf{H}(\mathbf{r}) &= \frac{4\pi}{c}\mathbf{J}(\mathbf{r}) - i\frac{\omega}{c}\mathbf{E}(\mathbf{r}).
\end{aligned} \tag{1}$$

In Eq. (1), $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ are, respectively, the space-dependent parts of the time-harmonic electric and magnetic fields $\mathcal{E}(\mathbf{r},t)=\Re\{\mathbf{E}(\mathbf{r})e^{-i\omega t}\}$ and $\mathcal{H}(\mathbf{r},t)=\Re\{\mathbf{H}(\mathbf{r})e^{-i\omega t}\}$, where $\Re$ denotes the real part; $\mathbf{r}$ and t denote the position and time, respectively; and ω is the frequency of oscillation. In addition, c is the speed of light in vacuum and

$$\rho(\mathbf{r}) = \nabla\cdot\mathbf{J}(\mathbf{r})/(i\omega) \tag{2}$$

and $\mathbf{J}(\mathbf{r})$ are, respectively, the space-dependent parts of the time-harmonic charge and current distributions $q(\mathbf{r},t)=\Re\{\rho(\mathbf{r})e^{-i\omega t}\}$ and $\mathcal{J}(\mathbf{r},t)=\Re\{\mathbf{J}(\mathbf{r})e^{-i\omega t}\}$. For the sake of brevity, we shall refer henceforth to the space-dependent parts $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ of the electric and magnetic fields $\mathcal{E}(\mathbf{r},t)$ and $\mathcal{H}(\mathbf{r},t)$, respectively, as ''the electric and magnetic fields.'' Similarly, we shall refer to $\mathbf{J}(\mathbf{r})$ as ''the current distribution.''

It is a well established fact (see, e.g., Refs. 15 and 23) that any NR current distribution $\mathbf{J}_{\rm NR}(\mathbf{r})$ of compact support σ admits the representation (henceforth to be referred to as ''the Devaney–Wolf representation'')

$$\mathbf{J}_{\rm NR}(\mathbf{r}) = \frac{1}{4\pi i}\left(\frac{c}{k}\right)[\nabla\times\nabla\times\mathbf{E}_{\rm NR}(\mathbf{r}) - k^2\mathbf{E}_{\rm NR}(\mathbf{r})], \tag{3}$$

where $\mathbf{E}_{\rm NR}(\mathbf{r})$ is a vector field of compact support σ. Furthermore, with every NR current distribution $\mathbf{J}_{\rm NR}(\mathbf{r})$ there is associated one and only one such field $\mathbf{E}_{\rm NR}(\mathbf{r})$ and this field is precisely the electric field produced by the NR current distribution[15] (see also Ref. 23, pp. 1107–1108). The scalar counterpart of the Devaney–Wolf representation appears to have been derived first by Friedlander[26] and has been used extensively in inverse source/inverse scattering theory.[8,10,11,23]

To simplify the following analysis, in the remainder of this paper we will restrict our attention to continuous NR electric and magnetic fields that are confined within a spherical volume $D{:}r \leqslant a$ of radius $a>0$, and that possess continuous curl and divergence on the boundary $r=a$ of the volume D. We shall refer henceforth to NR fields obeying all of these properties as ''well behaved NR fields.'' Well behaved NR fields are seen to obey, in view of the Maxwell equations (1), the following overspecified boundary conditions:

$$\begin{gathered}\mathbf{E}_{\rm NR}(\mathbf{r})|_{r=a}=0,\\ \nabla\times\mathbf{E}_{\rm NR}(\mathbf{r})|_{r=a}=0,\\ \nabla\cdot\mathbf{E}_{\rm NR}(\mathbf{r})|_{r=a}=0,\\ \nabla\times\nabla\times\mathbf{E}_{\rm NR}(\mathbf{r})|_{r=a}=0.\end{gathered} \tag{4}$$

The third and fourth conditions of Eq. (4) force the charge and current distributions associated with well behaved NR fields to vanish on the boundary $r=a$ of D. They thus ensure that the associated NR charge and current distributions will possess compact support in D. The purpose of Sec. III is to characterize all well behaved NR fields using (1) the vector spherical harmonics

$$\begin{gathered}\mathbf{P}_\ell^m(\theta,\phi) = \hat{\mathbf{r}}Y_\ell^m(\theta,\phi),\\ \mathbf{B}_\ell^m(\theta,\phi) = \frac{1}{\sqrt{\ell(\ell+1)}}\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi),\\ \mathbf{C}_\ell^m(\theta,\phi) = \frac{1}{\sqrt{\ell(\ell+1)}}\mathbf{L}Y_\ell^m(\theta,\phi),\end{gathered} \tag{5}$$

where $Y_\ell^m(\theta,\phi)$ is the spherical harmonic of degree ℓ and order m (as defined in Ref. 25, pp. 98–99) and $\mathbf{L}=-i\mathbf{r}\times\nabla$ is the orbital angular momentum operator [see, e.g., Ref. 25, Eq. (16.25)], (2) a Fourier–Bessel series, and (3) the overspecified boundary conditions (4). On the other hand, the goal of Sec. IV is to characterize all NR sources associated with well behaved NR fields by making use of the Devaney–Wolf representation Eq. (3).

III. A PROCEDURE FOR SPECIFYING NR FIELDS

This section provides a new procedure for specifying well behaved NR electric fields. Methodologically, we use a spherical vector function expansion to represent any continuous vector function that is confined within the spherical volume $D{:}r\leqslant a$ and vanishes on the boundary $r=a$ of D. Later we impose the additional constraints $\nabla\times\mathbf{E}_{\rm NR}(\mathbf{r})|_{r=a}=0$ and $\nabla\cdot\mathbf{E}_{\rm NR}(\mathbf{r})|_{r=a}=0$.

We have the following theorem.

Theorem 1: Any continuous vector function $\mathbf{F}(\mathbf{r})$ that is confined within the spherical volume $D{:}r\leqslant a$ and vanishes on the boundary $r=a$ of D can be represented, for $r\leqslant a$, in the form

$$\mathbf{F}(\mathbf{r})=\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}[a(n,\ell,m;\nu)\mathbf{P}_{\ell}^{m}(\theta,\phi)+b(n,\ell,m;\nu)\mathbf{B}_{\ell}^{m}(\theta,\phi)$$
$$+c(n,\ell,m;\nu)\mathbf{C}_{\ell}^{m}(\theta,\phi)]\psi_{n;\nu}(r), \tag{6}$$

where $\mathbf{P}_{\ell}^{m}(\theta,\phi)$, $\mathbf{B}_{\ell}^{m}(\theta,\phi)$, and $\mathbf{C}_{\ell}^{m}(\theta,\phi)$ are defined in Eq. (5) and

$$\psi_{n;\nu}(r)=\frac{\sqrt{2/a^3}}{|j_{\nu+1}(\beta_{\nu,n})|}j_{\nu}\left(\beta_{\nu,n}\frac{r}{a}\right), \tag{7}$$

where

$$j_{\nu}(x)=\sqrt{\frac{\pi}{2x}}J_{\nu+1/2}(x) \tag{8}$$

is the spherical Bessel function of the first kind of order ν, where ν is an arbitrary non-negative integer. The parameters $\beta_{\nu,n}$ are consecutive zeros of $j_{\nu}(x)$, i.e., $j_{\nu}(\beta_{\nu,n})=0$, $n=0,1,2,\ldots$. The expansion coefficients $a(n,\ell,m;\nu)$, $b(n,\ell,m;\nu)$, and $c(n,\ell,m;\nu)$ are given by

$$a(n,\ell,m;\nu)=\int_{r\leqslant a}dr\,r^2\psi_{n;\nu}(r)\int_{4\pi}d\Omega\,\mathbf{P}_{\ell}^{m*}(\theta,\phi)\cdot\mathbf{F}(\mathbf{r})$$
$$=\int_{r\leqslant a}dr\,r^2\psi_{n;\nu}(r)\int_{4\pi}d\Omega\,Y_{\ell}^{m*}(\theta,\phi)[\hat{\mathbf{r}}\cdot\mathbf{F}(\mathbf{r})], \tag{9}$$

$$b(n,\ell,m;\nu)=\int_{r\leqslant a}dr\,r^2\psi_{n;\nu}(r)\int_{4\pi}d\Omega\,\mathbf{B}_{\ell}^{m*}(\theta,\phi)\cdot\mathbf{F}(\mathbf{r})$$
$$=\frac{1}{\sqrt{\ell(\ell+1)}}\int_{r\leqslant a}dr\,r^2\psi_{n;\nu}(r)\int_{4\pi}d\Omega[\hat{\mathbf{r}}\times\mathbf{L}Y_{\ell}^{m}(\theta,\phi)]^*\cdot\mathbf{F}(\mathbf{r}) \tag{10}$$

and

$$c(n,\ell,m;\nu)=\int_{r\leqslant a}dr\,r^2\psi_{n;\nu}(r)\int_{4\pi}d\Omega\,\mathbf{C}_{\ell}^{m*}(\theta,\phi)\cdot\mathbf{F}(\mathbf{r})$$
$$=\frac{1}{\sqrt{\ell(\ell+1)}}\int_{r\leqslant a}dr\,r^2\psi_{n;\nu}(r)\int_{4\pi}d\Omega[\mathbf{L}Y_{\ell}^{m}(\theta,\phi)]^*\cdot\mathbf{F}(\mathbf{r}), \tag{11}$$

where $d\Omega=\sin\theta d\theta d\phi$ and an asterisk denotes the complex conjugate.

Proof: The proof of this result is straightforward and will not be given in detail. The key ingredients of the proof are (1) the completeness and orthogonality of the vector functions $\mathbf{P}_{\ell}^{m}(\theta,\phi)$, $\mathbf{B}_{\ell}^{m}(\theta,\phi)$, and $\mathbf{C}_{\ell}^{m}(\theta,\phi)$ over the unit sphere (see, e.g., Ref. 27, pp. 1898–1900); (2) the Fourier–Bessel series, which one can use to represent any function of r defined over the interval $[0,a]$ that is at least piecewise continuous and vanishes at $r=a$ (see, e.g., Eq. (11.51) in Ref. 28); and (3) the orthogonality property of the set of ordinary Bessel functions $J_{\nu}(\beta_{\nu,n}(r/a))$ for fixed non-negative integer ν and variable index n in the r interval $[0,a]$ [see, e.g., Eq. (11.168) in Ref. 28]. The latter property ensures that [see, e.g., Eq. (11.169) in Ref. 28]

$$\int_{r\leqslant a}dr\,r^2\psi_{n;\nu}(r)\psi_{n';\nu}(r)=\delta_{n,n'}, \tag{12}$$

where $\delta_{n,n'}$ is the Kronecker delta. The vector functions $\mathbf{P}_{\ell}^{m}(\theta,\phi)$, $\mathbf{B}_{\ell}^{m}(\theta,\phi)$, and $\mathbf{C}_{\ell}^{m}(\theta,\phi)$ are mutually perpendicular in view of the property $\mathbf{r}\cdot\mathbf{L}=0$ [see, e.g., Eq. (16.27) in Ref. 25]. They obey the orthogonality conditions

$$\int_{4\pi} d\Omega\,\mathbf{P}_{\ell}^{m*}(\theta,\phi)\cdot\mathbf{P}_{\ell'}^{m'}(\theta,\phi)=\delta_{\ell,\ell'}\delta_{m,m'},$$

$$\int_{4\pi} d\Omega\,\mathbf{B}_{\ell}^{m*}(\theta,\phi)\cdot\mathbf{B}_{\ell'}^{m'}(\theta,\phi)=\delta_{\ell,\ell'}\delta_{m,m'}, \tag{13}$$

$$\int_{4\pi} d\Omega\,\mathbf{C}_{\ell}^{m*}(\theta,\phi)\cdot\mathbf{C}_{\ell'}^{m'}(\theta,\phi)=\delta_{\ell,\ell'}\delta_{m,m'}.$$

Equation (13) follows from $L^2Y_{\ell}^{m}(\theta,\phi)=\ell(\ell+1)Y_{\ell}^{m}(\theta,\phi)$ [see, e.g., Eq. (16.24) in Ref. 25]. Also, $\mathbf{P}_0^0(\theta,\phi)=1/\sqrt{4\pi}\hat{\mathbf{r}}$ while $\mathbf{B}_0^0(\theta,\phi)=0$ and $\mathbf{C}_0^0(\theta,\phi)=0$.

The following result follows immediately from Theorem 1.

Theorem 2: Any well behaved NR electric field $\mathbf{E}_{\mathrm{NR}}(\mathbf{r})$ admits a representation of the form Eq. (6) [i.e., with $\mathbf{F}(\mathbf{r})$ substituted by $\mathbf{E}_{\mathrm{NR}}(\mathbf{r})$] subject to the constraints

$$\sum_{n=0}^{\infty} a(n,\ell,m;\nu)\alpha(n;\nu)=0,$$

$$\sum_{n=0}^{\infty} b(n,\ell,m;\nu)\alpha(n;\nu)=0, \tag{14}$$

$$\sum_{n=0}^{\infty} c(n,\ell,m;\nu)\alpha(n;\nu)=0,$$

where

$$\alpha(n;\nu)=\frac{1}{|j_{\nu+1}(\beta_{\nu,n})|}\frac{d}{dr}j_{\nu}\left(\beta_{\nu,n}\frac{r}{a}\right)\Big|_{r=a}.$$

Proof: That $\mathbf{E}_{\mathrm{NR}}(\mathbf{r})$ is representable in the form Eq. (6) follows from Theorem 1 and the above-imposed restrictions on $\mathbf{E}_{\mathrm{NR}}(\mathbf{r})$. After evaluating $\nabla\times\mathbf{E}_{\mathrm{NR}}(\mathbf{r})$ with $\mathbf{E}_{\mathrm{NR}}(\mathbf{r})$ given by the representation Eq. (6), with $\mathbf{F}(\mathbf{r})=\mathbf{E}_{\mathrm{NR}}(\mathbf{r})$, we obtain, by enforcing the condition $\nabla\times\mathbf{E}_{\mathrm{NR}}(\mathbf{r})|_{r=a}=0$, the result

$$\sum_{n=0}^{\infty}\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}[c(n,\ell,m;\nu)\alpha(n;\nu)\mathbf{B}_{\ell}^{m}(\theta,\phi)-b(n,\ell,m;\nu)\alpha(n;\nu)\mathbf{C}_{\ell}^{m}(\theta,\phi)]=0, \tag{15}$$

where we have discarded unnecessary constants. In deriving Eq. (15) we have made use of the results (see Appendix A)

$$\nabla\times\left[j_{\nu}\left(\beta_{\nu,n}\frac{r}{a}\right)\hat{\mathbf{r}}Y_{\ell}^{m}(\theta,\phi)\right]=-\frac{i}{r}j_{\nu}\left(\beta_{\nu,n}\frac{r}{a}\right)\mathbf{L}Y_{\ell}^{m}(\theta,\phi),$$

$$\nabla\times\left[j_{\nu}\left(\beta_{\nu,n}\frac{r}{a}\right)\mathbf{L}Y_{\ell}^{m}(\theta,\phi)\right]=\frac{i\ell(\ell+1)}{r}j_{\nu}\left(\beta_{\nu,n}\frac{r}{a}\right)\hat{\mathbf{r}}Y_{\ell}^{m}(\theta,\phi)+\frac{1}{r}\frac{d}{dr}\left[rj_{\nu}\left(\beta_{\nu,n}\frac{r}{a}\right)\right]\hat{\mathbf{r}}\times\mathbf{L}Y_{\ell}^{m}(\theta,\phi), \tag{16}$$

$$\nabla\times\left[j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi)\right]=-\frac{1}{r}\frac{d}{dr}\left[rj_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\right]\mathbf{L}Y_\ell^m(\theta,\phi).$$

Similarly, by evaluating $\nabla\cdot\mathbf{E}_{\mathrm{NR}}(\mathbf{r})$ with $\mathbf{E}_{\mathrm{NR}}(\mathbf{r})$ given by Eq. (6), with $\mathbf{F}(\mathbf{r})=\mathbf{E}_{\mathrm{NR}}(\mathbf{r})$, while enforcing the condition $\nabla\cdot\mathbf{E}_{\mathrm{NR}}(\mathbf{r})|_{r=a}=0$, one obtains

$$\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}a(n,\ell,m;\nu)\alpha(n;\nu)Y_\ell^m(\theta,\phi)=0. \tag{17}$$

In deriving Eq. (17) we have made use of the results (see Appendix A)

$$\nabla\cdot\left[j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\hat{\mathbf{r}}Y_\ell^m(\theta,\phi)\right]=\left[\frac{2}{r}j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)+\frac{d}{dr}j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\right]Y_\ell^m(\theta,\phi),$$

$$\nabla\cdot\left[j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\mathbf{L}Y_\ell^m(\theta,\phi)\right]=0, \tag{18}$$

$$\nabla\cdot\left[j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi)\right]=-\frac{i\ell(\ell+1)}{r}j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)Y_\ell^m(\theta,\phi).$$

Finally, Eq. (14) follows from Eqs. (15) and (17) and the orthogonality relations (13). The fourth of the overspecified boundary conditions Eq. (4), i.e., $\nabla\times\nabla\times\mathbf{E}_{\mathrm{NR}}(\mathbf{r})|_{r=a}=0$, is automatically satisfied so long as Eq. (14) holds, as we will see in Sec. IV.

Now, ν is an arbitrary non-negative integer. In the remainder of the paper we will restrict our analysis to the special case $\nu=0$, although the general theory applies to arbitrary non-negative integers ν.

A. Special case: $\nu=0$

For $\nu=0$ we obtain from Theorem 2

$$\mathbf{E}_{\mathrm{NR}}(\mathbf{r})=\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}[a(n,\ell,m)\mathbf{P}_\ell^m(\theta,\phi)+b(n,\ell,m)\mathbf{B}_\ell^m(\theta,\phi)+c(n,\ell,m)\mathbf{C}_\ell^m(\theta,\phi)]\psi_n(r), \tag{19}$$

where we have defined

$$a(n,\ell,m)=a(n,\ell,m;\nu=0),$$
$$b(n,\ell,m)=b(n,\ell,m;\nu=0), \tag{20}$$
$$c(n,\ell,m)=c(n,\ell,m;\nu=0),$$

$$\psi_n(r)=\psi_{n;\nu=0}(r)=\sqrt{2/a^3}\beta_{0,n}j_0\left(\beta_{0,n}\frac{r}{a}\right).$$

For this special case we obtain

$$\beta_{0,n}=(n+1)\pi, \tag{21}$$

$$j_0\left(\beta_{0,n}\frac{r}{a}\right)=\frac{\sin\left(\beta_{0,n}\frac{r}{a}\right)}{\beta_{0,n}\frac{r}{a}} \tag{22}$$

and the constraint relations (14) reduce to

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)a(n,\ell,m)&=0,\\ \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)b(n,\ell,m)&=0,\\ \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)c(n,\ell,m)&=0,\end{aligned} \tag{23}$$

where we have discarded unnecessary constants.

B. An orthonormal basis for NR fields

We can now use the results established in Sec. III A to generate an orthonormal basis for all well behaved NR electric fields. Following the vector counterpart of the procedure used in Ref. 24 for scalar fields, we define the three sequences of NR electric fields $\{\mathbf{E}_a^{(p,\ell,m)}(\mathbf{r})\}$, $\{\mathbf{E}_b^{(p,\ell,m)}(\mathbf{r})\}$, and $\{\mathbf{E}_c^{(p,\ell,m)}(\mathbf{r})\}$, with $p=1,2,\ldots$; $\ell=0,1,\ldots$; and $m=-\ell,-\ell+1,\ldots,\ell$, where

$$\begin{aligned}\mathbf{E}_a^{(p,\ell,m)}(\mathbf{r})&=\mathbf{P}_\ell^m(\theta,\phi)\sum_{n=0}^{p}v_a^{(p)}(n)\psi_n(r),\\ \mathbf{E}_b^{(p,\ell,m)}(\mathbf{r})&=\mathbf{B}_\ell^m(\theta,\phi)\sum_{n=0}^{p}v_b^{(p)}(n)\psi_n(r),\\ \mathbf{E}_c^{(p,\ell,m)}(\mathbf{r})&=\mathbf{C}_\ell^m(\theta,\phi)\sum_{n=0}^{p}v_c^{(p)}(n)\psi_n(r),\end{aligned} \tag{24}$$

where the expansion coefficients $v_a^{(p)}(n)$, $v_b^{(p)}(n)$, and $v_c^{(p)}(n)$ must obey, by analogy with Eq. (23), the constraint equations

$$\sum_{n=0}^{p}(-1)^{n+1}(n+1)v_j^{(p)}(n)=0,\quad j=a,b,c. \tag{25}$$

Next, we impose the orthonormality conditions

$$\int_D d^3r\,\mathbf{E}_j^{(p,\ell,m)*}(\mathbf{r})\cdot\mathbf{E}_j^{(p',\ell',m')}(\mathbf{r})=\delta_{\ell,\ell'}\delta_{m,m'}\delta_{p,p'},\quad j=a,b,c. \tag{26}$$

In view of Eq. (12), Eq. (13), and Eq. (24), the condition Eq. (26) yields

$$\sum_{n=0}^{p}v_j^{(p)*}(n)v_j^{(p')}(n)=\delta_{p,p'},\quad j=a,b,c. \tag{27}$$

It follows at once from Theorem 2 and the definitions and constraints for $\mathbf{E}_a^{(p,\ell,m)}(\mathbf{r})$, $\mathbf{E}_b^{(p,\ell,m)}(\mathbf{r})$, and $\mathbf{E}_c^{(p,\ell,m)}(\mathbf{r})$ above that any well behaved NR electric field $\mathbf{E}_{\rm NR}(\mathbf{r})$ can be written as

$$\mathbf{E}_{\rm NR}(\mathbf{r})=\sum_{p=1}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}[u_a(p,\ell,m)\mathbf{E}_a^{(p,\ell,m)}(\mathbf{r})+u_b(p,\ell,m)\mathbf{E}_b^{(p,\ell,m)}(\mathbf{r})+u_c(p,\ell,m)\mathbf{E}_c^{(p,\ell,m)}(\mathbf{r})], \tag{28}$$

where

$$u_j(p,\ell,m)=\int_D d^3r\mathbf{E}_j^{(p,\ell,m)*}(\mathbf{r})\cdot\mathbf{E}_{\rm NR}(\mathbf{r}),\quad j=a,b,c. \tag{29}$$

Thus, the vector functions $\mathbf{E}_a^{(p,\ell,m)}(\mathbf{r})$, $\mathbf{E}_b^{(p,\ell,m)}(\mathbf{r})$, and $\mathbf{E}_c^{(p,\ell,m)}(\mathbf{r})$, with $p=1,2,\ldots;$ $\ell=0,1,\ldots;$ and $m=-\ell,-\ell+1,\ldots,\ell$, form an orthonormal basis for all well behaved NR electric fields, so long as the expansion coefficients $v_j^{(p,\ell,m)}(n,\ell,m)$ in Eq. (24) satisfy the constraint relations (25) and the orthonormality conditions (27).

Now we note that Eqs. (25) and (27) can be jointly satisfied, as follows from the fact that each basis field, $\mathbf{E}_j^{(p,\ell,m)}(\mathbf{r})$, $j=a,b,c$, is defined from Eq. (24) by a sum of $p+1$ linearly independent functions, while condition (27) only involves the first p_0+1 of these functions, where p_0 is the lower of p,p'. This consideration leads to the following procedure for constructing the orthonormal set. The basis fields $\mathbf{E}_j^{(1,\ell,m)}(\mathbf{r})$, $j=a,b,c$, are constructed with $v_j^{(1)}(0)$ and $v_j^{(1)}(1)$ chosen so as to obey conditions (25) and (27) with $p=p'=1$. The basis fields $\mathbf{E}_j^{(2,\ell,m)}(\mathbf{r})$, $j=a,b,c$, are constructed with $v_j^{(2)}(0)$ and $v_j^{(2)}(1)$ selected so as to satisfy Eq. (27) with $p=1$ and $p'=2$. This leaves $v_j^{(2)}(2)$ arbitrary and also leaves $v_j^{(2)}(0)$ and $v_j^{(2)}(1)$ arbitrary up to a single multiplicative constant. The multiplicative constant and $v_j^{(2)}(2)$ are then uniquely determined from the constraint equation (25) and the orthonormality condition (27) with $p=2$ and $p'=2$. The above-outlined step-by-step procedure is elaborated in Appendix B and can be used to construct the remaining basis fields $\mathbf{E}_j^{(p,\ell,m)}(\mathbf{r})$, $j=a,b,c$, i.e., those corresponding to $p>2$. By means of this procedure we have found the coefficients $v_j^{(p)}(n)$ to be defined, for arbitrary $p=1,2,\ldots$, by the expressions

$$v_j^{(p)}(0)=\left\{\sum_{n=0}^{p-1}(n+1)^2+\frac{[\Sigma_{n=0}^{p-1}(n+1)^2]^2}{(p+1)^2}\right\}^{-1/2}, \tag{30}$$

$$v_j^{(p)}(n)=(-1)^n(n+1)v_j^{(p)}(0),0<n<p, \tag{31}$$

and

$$v_j^{(p)}(p)=v_j^{(p)}(0)\frac{(-1)^{p+1}\Sigma_{n=0}^{p-1}(n+1)^2}{p+1}. \tag{32}$$

Finally, we note that the expansion coefficients $v_j^{(p)}(n)$ obey, in view of Eqs. (25) and (27), the same constraint equations for $i=a$, b, and c. This enables us to use Eqs. (30)–(32) to express Eq. (24) in the convenient form

$$\begin{aligned}\mathbf{E}_a^{(p,\ell,m)}(\mathbf{r})&=F_p(r)\mathbf{P}_\ell^m(\theta,\phi),\\ \mathbf{E}_b^{(p,\ell,m)}(\mathbf{r})&=F_p(r)\mathbf{B}_\ell^m(\theta,\phi),\\ \mathbf{E}_c^{(p,\ell,m)}(\mathbf{r})&=F_p(r)\mathbf{C}_\ell^m(\theta,\phi),\end{aligned} \tag{33}$$

where

$$F_p(r)=\sum_{n=0}^{p} v_j^{(p)}(n)\psi_n(r), \quad j=a,b,c$$
$$=\left\{\sum_{n=0}^{p-1}(n+1)^2+\frac{[\Sigma_{n=0}^{p-1}(n+1)^2]^2}{(p+1)^2}\right\}^{-1/2}$$
$$\times\left\{\left[\sum\nolimits_{n=0}^{p-1}(-1)^n(n+1)\psi_n(r)\right]+(-1)^{p+1}\frac{\Sigma_{n=0}^{p-1}(n+1)^2}{p+1}\psi_p(r)\right\}. \tag{34}$$

Thus, we obtain from Eq. (34) and Eqs. (20) and (21)

$$F_1(r)=\frac{2\sqrt{2}\pi}{\sqrt{5a^3}}\left[j_0\left(\pi\frac{r}{a}\right)+j_0\left(2\pi\frac{r}{a}\right)\right],$$

$$F_2(r)=\frac{3\pi}{\sqrt{35a^3}}\left[j_0\left(\pi\frac{r}{a}\right)-4j_0\left(2\pi\frac{r}{a}\right)-5j_0\left(3\pi\frac{r}{a}\right)\right],$$

$$F_3(r)=\frac{2\sqrt{2}\pi}{\sqrt{105a^3}}\left[j_0\left(\pi\frac{r}{a}\right)-4j_0\left(2\pi\frac{r}{a}\right)+9j_0\left(3\pi\frac{r}{a}\right)+14j_0\left(4\pi\frac{r}{a}\right)\right], \tag{35}$$

$$F_4(r)=\frac{\pi}{\sqrt{33a^3}}\left[j_0\left(\pi\frac{r}{a}\right)-4j_0\left(2\pi\frac{r}{a}\right)+9j_0\left(3\pi\frac{r}{a}\right)-16j_0\left(4\pi\frac{r}{a}\right)-30j_0\left(5\pi\frac{r}{a}\right)\right],$$

$$F_5(r)=\frac{6\sqrt{2}\pi}{\sqrt{5005a^3}}\left[j_0\left(\pi\frac{r}{a}\right)-4j_0\left(2\pi\frac{r}{a}\right)+9j_0\left(3\pi\frac{r}{a}\right)-16j_0\left(4\pi\frac{r}{a}\right)\right.$$
$$\left.+25j_0\left(5\pi\frac{r}{a}\right)+55j_0\left(6\pi\frac{r}{a}\right)\right]$$

and so on. The result Eq. (34) is of great value since it gives explicit form to the orthonormal set.

IV. A PROCEDURE FOR SPECIFYING NR CURRENT DISTRIBUTIONS

In this section we make use of the Devaney–Wolf representation Eq. (3) and the results of Sec. III to characterize all NR current distributions associated with well behaved NR fields. By using the representation Eq. (3) with $\mathbf{E}_{\rm NR}(\mathbf{r})$ given by Eqs. (19)–(22) subject to the constraint conditions (23), we obtain, for $r\leqslant a$, the following representation for the NR current distributions associated with the NR electric fields in Sec. III:

$$\mathbf{J}_{\rm NR}(\mathbf{r})=\frac{1}{4\pi i}\left(\frac{c}{k}\right)[\nabla\times\nabla\times\mathbf{E}_{\rm NR}(\mathbf{r})-k^2\mathbf{E}_{\rm NR}(\mathbf{r})]$$
$$=\frac{1}{4\pi i}\left(\frac{c}{k}\right)\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}[R_a^{(n,\ell,m)}(r)\mathbf{P}_\ell^m(\theta,\phi)+R_b^{(n,\ell,m)}(r)\mathbf{B}_\ell^m(\theta,\phi)$$
$$+R_c^{(n,\ell,m)}(r)\mathbf{C}_\ell^m(\theta,\phi)], \tag{36}$$

where

$$R_a^{(n,\ell,m)}(r)=\sqrt{2/a^3}\beta_{0,n}\left\{a(n,\ell,m)\left[\frac{\ell(\ell+1)}{r^2}-k^2\right]j_0\left(\beta_{0,n}\frac{r}{a}\right)\right.$$
$$\left.-b(n,\ell,m)\frac{i\sqrt{\ell(\ell+1)}}{r^2}\frac{d}{dr}\left[rj_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\right\}, \quad (37)$$

$$R_b^{(n,\ell,m)}(r)=\sqrt{2/a^3}\beta_{0,n}\left\{b(n,\ell,m)\left[\left[\frac{(n+1)\pi}{a}\right]^2-k^2\right]j_0\left(\beta_{0,n}\frac{r}{a}\right)\right.$$
$$\left.-a(n,\ell,m)\frac{i\sqrt{\ell(\ell+1)}}{r}\frac{d}{dr}j_0\left(\beta_{0,n}\frac{r}{a}\right)\right\}, \quad (38)$$

and

$$R_c^{(n,\ell,m)}(r)=\sqrt{2/a^3}\beta_{0,n}c(n,\ell,m)j_0\left(\beta_{0,n}\frac{r}{a}\right)\left\{\left[\frac{(n+1)\pi}{a}\right]^2+\frac{\ell(\ell+1)}{r^2}-k^2\right\}. \quad (39)$$

In deriving Eqs. (38) and (39) we have made use of the fact that[24]

$$\nabla^2 j_0\left(\beta_{0,n}\frac{r}{a}\right)=\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}j_0\left(\beta_{0,n}\frac{r}{a}\right)\right]=-\left[\frac{(n+1)\pi}{a}\right]^2 j_0\left(\beta_{0,n}\frac{r}{a}\right) \quad (40)$$

and

$$\nabla^2 Y_\ell^m(\theta,\phi)=-\frac{L^2}{r^2}Y_\ell^m(\theta,\phi)=-\frac{\ell(\ell+1)}{r^2}Y_\ell^m(\theta,\phi). \quad (41)$$

Also, in carrying out the manipulations leading to Eqs. (37)–(39) we have made use of the following results (see Appendix C):

$$\nabla\times\nabla\times\left[j_0\left(\beta_{0,n}\frac{r}{a}\right)\hat{\mathbf{r}}Y_\ell^m(\theta,\phi)\right]=\frac{\ell(\ell+1)}{r^2}j_0\left(\beta_{0,n}\frac{r}{a}\right)\hat{\mathbf{r}}Y_\ell^m(\theta,\phi)-\frac{i}{r}\frac{d}{dr}j_0\left(\beta_{0,n}\frac{r}{a}\right)\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi),$$

$$\nabla\times\nabla\times\left[j_0\left(\beta_{0,n}\frac{r}{a}\right)\mathbf{L}Y_\ell^m(\theta,\phi)\right]=\left\{\left[\frac{(n+1)\pi}{a}\right]^2+\frac{\ell(\ell+1)}{r^2}\right\}j_0\left(\beta_{0,n}\frac{r}{a}\right)\mathbf{L}Y_\ell^m(\theta,\phi), \quad (42)$$

$$\nabla\times\nabla\times\left[j_0\left(\beta_{0,n}\frac{r}{a}\right)\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi)\right]=-\frac{i\ell(\ell+1)}{r^2}\frac{d}{dr}\left[rj_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\hat{\mathbf{r}}Y_\ell^m(\theta,\phi)$$
$$+\left[\frac{(n+1)\pi}{a}\right]^2 j_0\left(\beta_{0,n}\frac{r}{a}\right)\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi).$$

The charge density $\rho_{\mathrm{NR}}(\mathbf{r})$ corresponding to the NR current distribution $\mathbf{J}_{\mathrm{NR}}(\mathbf{r})$ in Eq. (36) is evaluated by using the procedure employed to derive Eq. (18) in Appendix A. We obtain from Eqs. (36) to (39)

$$\rho_{\mathrm{NR}}(\mathbf{r})=\nabla\cdot\mathbf{J}_{\mathrm{NR}}(\mathbf{r})/(i\omega)$$
$$=-\frac{1}{4\pi\omega}\left(\frac{c}{k}\right)\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\frac{1}{r^2}\frac{d}{dr}[r^2R_a^{(n,\ell,m)}(r)]-\frac{i\ell(\ell+1)}{r}R_b^{(n,\ell,m)}(r)\right\}Y_\ell^m(\theta,\phi), \tag{43}$$

where

$$\frac{1}{r^2}\frac{d}{dr}[r^2R_a^{(n,\ell,m)}(r)]=\sqrt{2/a^3}\beta_{0,n}\left\{a(n,\ell,m)\left[\left[\frac{\ell(\ell+1)}{r^2}-k^2\right]\frac{d}{dr}j_0\left(\beta_{0,n}\frac{r}{a}\right)-\frac{2k^2}{r}j_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\right.$$
$$\left.-b(n,\ell,m)\frac{i\sqrt{\ell(\ell+1)}}{r^2}\frac{d^2}{dr^2}\left[rj_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\right\}. \tag{44}$$

By referring to the constraint conditions (23), $\mathbf{J}_{\mathrm{NR}}(\mathbf{r})$ and $\rho_{\mathrm{NR}}(\mathbf{r})$, defined by Eqs. (36)–(39) and Eqs. (43) and (44), respectively, can be shown to vanish on the boundary $r=a$ of D, as expected from Eqs. (1) and (4).

We can apply now a procedure analogous to that used in Sec. III B to generate a (nonorthogonal) basis for NR current distributions confined within D. Thus, we build the three sequences of NR current distributions $\{\mathbf{J}_a^{(p,\ell,m)}(\mathbf{r})\}$, $\{\mathbf{J}_b^{(p,\ell,m)}(\mathbf{r})\}$, and $\{\mathbf{J}_c^{(p,\ell,m)}(\mathbf{r})\}$ associated with the sequences of NR fields $\{\mathbf{E}_a^{(p,\ell,m)}(\mathbf{r})\}$, $\{\mathbf{E}_b^{(p,\ell,m)}(\mathbf{r})\}$, and $\{\mathbf{E}_c^{(p,\ell,m)}(\mathbf{r})\}$, respectively, with $p=1,2,\ldots$; $\ell=0,1,\ldots$; and $m=-\ell,-\ell+1,\ldots,\ell$, where

$$\mathbf{J}_a^{(p,\ell,m)}(\mathbf{r})=\frac{1}{4\pi i}\left(\frac{c}{k}\right)[\nabla\times\nabla\times\mathbf{E}_a^{(p,\ell,m)}(\mathbf{r})-k^2\mathbf{E}_a^{(p,\ell,m)}(\mathbf{r})]$$
$$=\frac{1}{4\pi i}\left(\frac{c}{k}\right)\sqrt{2/a^3}\sum_{n=0}^{p}v_a^{(p)}(n)\beta_{0,n}\left\{\left[\frac{\ell(\ell+1)}{r^2}-k^2\right]j_0\left(\beta_{0,n}\frac{r}{a}\right)\mathbf{P}_\ell^m(\theta,\phi)\right.$$
$$\left.-\frac{i\sqrt{\ell(\ell+1)}}{r}\frac{d}{dr}j_0\left(\beta_{0,n}\frac{r}{a}\right)\mathbf{B}_\ell^m(\theta,\phi)\right\}, \tag{45}$$

$$\mathbf{J}_b^{(p,\ell,m)}(\mathbf{r})=\frac{1}{4\pi i}\left(\frac{c}{k}\right)[\nabla\times\nabla\times\mathbf{E}_b^{(p,\ell,m)}(\mathbf{r})-k^2\mathbf{E}_b^{(p,\ell,m)}(\mathbf{r})]$$
$$=\frac{1}{4\pi i}\left(\frac{c}{k}\right)\sqrt{2/a^3}\sum_{n=0}^{p}v_b^{(p)}(n)\beta_{0,n}\left\{-\frac{i\sqrt{\ell(\ell+1)}}{r^2}\frac{d}{dr}\left[rj_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\mathbf{P}_\ell^m(\theta,\phi)\right.$$
$$\left.+\left[\left[\frac{(n+1)\pi}{a}\right]^2-k^2\right]j_0\left(\beta_{0,n}\frac{r}{a}\right)\mathbf{B}_\ell^m(\theta,\phi)\right\} \tag{46}$$

and

$$\mathbf{J}_c^{(p,\ell,m)}(\mathbf{r})=\frac{1}{4\pi i}\left(\frac{c}{k}\right)[\nabla\times\nabla\times\mathbf{E}_c^{(p,\ell,m)}(\mathbf{r})-k^2\mathbf{E}_c^{(p,\ell,m)}(\mathbf{r})]$$
$$=\frac{1}{4\pi i}\left(\frac{c}{k}\right)\sqrt{2/a^3}\sum_{n=0}^{p}v_c^{(p)}(n)\beta_{0,n}\left[\left[\frac{(n+1)\pi}{a}\right]^2+\frac{\ell(\ell+1)}{r^2}-k^2\right]j_0\left(\beta_{0,n}\frac{r}{a}\right)\mathbf{C}_\ell^m(\theta,\phi). \tag{47}$$

In deriving Eqs. (45)–(47) we have made use of Eq. (42). The expansion coefficients $v_a^{(p)}(n)$, $v_b^{(p)}(n)$, and $v_c^{(p)}(n)$ are defined by Eqs. (30), (31), and (32).

Finally, by following a procedure analogous to that employed in deriving Eqs. (43) and (44), we find the charge densities $\rho_a^{(p,\ell,m)}(\mathbf{r})$, $\rho_b^{(p,\ell,m)}(\mathbf{r})$, and $\rho_c^{(p,\ell,m)}(\mathbf{r})$ associated with the basis NR current distributions $\mathbf{J}_a^{(p,\ell,m)}(\mathbf{r})$, $\mathbf{J}_b^{(p,\ell,m)}(\mathbf{r})$, and $\mathbf{J}_c^{(p,\ell,m)}(\mathbf{r})$, respectively, to be given from Eqs. (45), (46), and (47) by

$$\begin{aligned}\rho_a^{(p,\ell,m)}(\mathbf{r})&=\nabla\cdot\mathbf{J}_a^{(p,\ell,m)}(\mathbf{r})/(i\omega)\\&=\frac{1}{4\pi}\sqrt{2/a^3}\sum_{n=0}^{p} v_a^{(p)}(n)\beta_{0,n}\left[\frac{d}{dr}j_0\left(\beta_{0,n}\frac{r}{a}\right)+\frac{2}{r}j_0\left(\beta_{0,n}\frac{r}{a}\right)\right]Y_\ell^m(\theta,\phi),\end{aligned}\tag{48}$$

$$\begin{aligned}\rho_b^{(p,\ell,m)}(\mathbf{r})&=\nabla\cdot\mathbf{J}_b^{(p,\ell,m)}(\mathbf{r})/(i\omega)\\&=\frac{1}{4\pi\omega}\left(\frac{c}{k}\right)\sqrt{2/a^3}\sum_{n=0}^{p} v_b^{(p)}(n)\beta_{0,n}\left\{\frac{i\sqrt{\ell(\ell+1)}}{r^2}\frac{d^2}{dr^2}\left[rj_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\right.\\&\quad\left.+\frac{i\ell(\ell+1)}{r}\left[\left[\frac{(n+1)\pi}{a}\right]^2-k^2\right]j_0\left(\beta_{0,n}\frac{r}{a}\right)\right\}Y_\ell^m(\theta,\phi),\end{aligned}\tag{49}$$

and

$$\rho_c^{(p,\ell,m)}(\mathbf{r})=\nabla\cdot\mathbf{J}_c^{(p,\ell,m)}(\mathbf{r})/(i\omega)=0.\tag{50}$$

V. SPECIAL CASES

In this section we examine the two simplest classes of NR current distributions that can be constructed from the results of Secs. III and IV. They are: (1) spherically symmetric NR current distributions, and (2) NR current distributions with dipolar angular dependence.

A. Spherically symmetric NR current distributions (case $\ell=0$)

We consider next the simple example of spherically symmetric NR current distributions. Such NR current distributions are purely longitudinal, which automatically makes them NR. An example is provided by a spherically symmetric charge distribution undergoing oscillatory radial motion. Spherically symmetric NR current distributions can be constructed by using the basis functions $\mathbf{J}_a^{(p,0,0)}(\mathbf{r})$, $\mathbf{J}_b^{(p,0,0)}(\mathbf{r})$, and $\mathbf{J}_c^{(p,0,0)}(\mathbf{r})$, corresponding to the case $\ell=0$, $m=0$ in Eqs. (45), (46), and (47). In particular,

$$\mathbf{J}_a^{(p,0,0)}(\mathbf{r})=\frac{i\omega}{(4\pi)^{3/2}}\sqrt{2/a^3}\sum_{n=0}^{p} v_a^{(p)}(n)\beta_{0,n}j_0\left(\beta_{0,n}\frac{r}{a}\right)\hat{\mathbf{r}},\tag{51}$$

$\mathbf{J}_b^{(p,0,0)}(\mathbf{r})=0$ and $\mathbf{J}_c^{(p,0,0)}(\mathbf{r})=0$. The coefficients $v_a^{(p)}(n)$ in Eq. (51) are given by Eqs. (30)–(32) while $\beta_{0,n}$ is defined by Eq. (21). Then we obtain from Eq. (51)

$$\begin{aligned}\mathbf{J}_a^{(1,0,0)}(\mathbf{r})&=\frac{i\omega}{4}\sqrt{\frac{2}{5\pi a^3}}\left[j_0\left(\pi\frac{r}{a}\right)+j_0\left(2\pi\frac{r}{a}\right)\right]\hat{\mathbf{r}},\\\mathbf{J}_a^{(2,0,0)}(\mathbf{r})&=\frac{3i\omega}{8}\sqrt{\frac{1}{35\pi a^3}}\left[j_0\left(\pi\frac{r}{a}\right)-4j_0\left(2\pi\frac{r}{a}\right)-5j_0\left(3\pi\frac{r}{a}\right)\right]\hat{\mathbf{r}},\\\mathbf{J}_a^{(3,0,0)}(\mathbf{r})&=\frac{i\omega}{4}\sqrt{\frac{2}{105\pi a^3}}\left[j_0\left(\pi\frac{r}{a}\right)-4j_0\left(2\pi\frac{r}{a}\right)+9j_0\left(3\pi\frac{r}{a}\right)+14j_0\left(4\pi\frac{r}{a}\right)\right]\hat{\mathbf{r}},\end{aligned}\tag{52}$$

$$\mathbf{J}_a^{(4,0,0)}(\mathbf{r}) = \frac{i\omega}{8}\sqrt{\frac{1}{33\pi a^3}}\left[j_0\left(\pi\frac{r}{a}\right) - 4j_0\left(2\pi\frac{r}{a}\right) + 9j_0\left(3\pi\frac{r}{a}\right) - 16j_0\left(4\pi\frac{r}{a}\right) - 30j_0\left(5\pi\frac{r}{a}\right)\right]\hat{\mathbf{r}},$$

$$\mathbf{J}_a^{(5,0,0)}(\mathbf{r}) = \frac{3i\omega}{4}\sqrt{\frac{2}{5005\pi a^3}}\left[j_0\left(\pi\frac{r}{a}\right) - 4j_0\left(2\pi\frac{r}{a}\right) + 9j_0\left(3\pi\frac{r}{a}\right) - 16j_0\left(4\pi\frac{r}{a}\right) + 25j_0\left(5\pi\frac{r}{a}\right) + 55j_0\left(6\pi\frac{r}{a}\right)\right]\hat{\mathbf{r}},$$

and so on. On the other hand, the associated charge densities $\rho_a^{(p,0,0)}(\mathbf{r})$ are given from Eq. (48) by

$$\rho_a^{(p,0,0)}(\mathbf{r}) = \frac{1}{(4\pi)^{3/2}}\sqrt{2/a^3}\sum_{n=0}^{p} v_a^{(p)}(n)\beta_{0,n}\left[\frac{2}{r}j_0\left(\beta_{0,n}\frac{r}{a}\right) + \frac{d}{dr}j_0\left(\beta_{0,n}\frac{r}{a}\right)\right]. \tag{53}$$

Now we can represent any well behaved spherically symmetric NR current distribution $\mathbf{J}(\mathbf{r})$ as

$$\mathbf{J}(\mathbf{r}) = \sum_{p=0}^{\infty} q_a(p,0,0)\mathbf{J}_a^{(p,0,0)}(\mathbf{r}), \tag{54}$$

where

$$q_a(p,0,0) = \int_D d^3r\,\mathbf{J}_a^{(p,0,0)*}(\mathbf{r})\cdot\mathbf{J}(\mathbf{r}). \tag{55}$$

Finally, the NR fields $\mathbf{E}_a^{(p,0,0)}(\mathbf{r})$ produced by the NR current distributions $\mathbf{J}_a^{(p,0,0)}(\mathbf{r})$ are found from Eqs. (20), (24), and (51) to be given by

$$\mathbf{E}_a^{(p,0,0)}(\mathbf{r}) = -\frac{4\pi i}{\omega}\mathbf{J}_a^{(p,0,0)}(\mathbf{r}). \tag{56}$$

Spherically symmetric NR current distributions are then seen to be, apart from a proportionality factor, identical to the NR fields they produce. It is not hard to show that this applies to any time-harmonic longitudinal NR current distribution.

B. NR sources and fields with dipolar angular dependence (case $\ell=1$, $m=0$)

We consider next the case of NR sources and fields with dipolar angular dependence, i.e., NR sources and fields described by series expansions over $\mathbf{J}_c^{(p,1,0)}(\mathbf{r})$ and $\mathbf{E}_c^{(p,1,0)}(\mathbf{r})$, respectively, corresponding to the case $\ell=1$, $m=0$. Physically, NR current distributions of this kind are similar to a loop of current confined within a spherical region. These NR current distributions are formed by superposing certain radiating magnetic dipole-like sources in a way that makes their radiated fields cancel out for $r>a$ by destructive interference, as we shall show in the following. The electric counterpart of these NR collections of magnetic dipoles (i.e., NR collections of electric dipoles) can be built, by duality, using the results of this section.

In this case we obtain from the Devaney–Wolf representation Eq. (3) and Eqs. (28), (29), and (47)

$$\mathbf{E}_{\mathrm{NR}}(\mathbf{r}) = \sum_{p=0}^{\infty} u_c(p,1,0)\mathbf{E}_c^{(p,1,0)}(\mathbf{r}), \tag{57}$$

where

$$u_c(p,1,0)=\int_D d^3r\mathbf{E}_c^{(p,1,0)*}(\mathbf{r})\cdot\mathbf{E}_{\mathrm{NR}}(\mathbf{r}) \tag{58}$$

and

$$\mathbf{J}(\mathbf{r})=\sum_{p=0}^{\infty} q_c(p,1,0)\mathbf{J}_c^{(p,1,0)}(\mathbf{r}) \tag{59}$$

where

$$q_c(p,1,0)=\int_D d^3r\mathbf{J}_c^{(p,1,0)*}(\mathbf{r})\cdot\mathbf{J}(\mathbf{r}). \tag{60}$$

By using

$$\mathbf{L}Y_1^0(\hat{\mathbf{r}})=i\mathbf{u}_\phi\sqrt{\frac{3}{4\pi}}\sin\theta, \tag{61}$$

where $\mathbf{u}_\phi$ is the unit vector in the positive ϕ direction, the basis fields $\mathbf{E}_c^{(p,1,0)}(\mathbf{r})$ are found from Eqs. (33) and (61) to be given by

$$\mathbf{E}_c^{(p,1,0)}(\mathbf{r})=i\mathbf{u}_\phi\sqrt{\frac{3}{8\pi}}\sin\theta F_p(r), \tag{62}$$

where $F_p(r)$ is defined by Eq. (34). For example, for $p=1,2,\ldots,5$, the basis fields $\mathbf{E}_c^{(p,1,0)}(\mathbf{r})$ are given explicitly by Eqs. (35) and (62). On the other hand, the associated NR current distributions $\mathbf{J}_c^{(p,1,0)}(\mathbf{r})$, defined from Eq. (47) as

$$\mathbf{J}_c^{(p,1,0)}(\mathbf{r})=\frac{1}{4\pi i}\left(\frac{c}{ka^{3/2}}\right)\mathbf{L}Y_1^0(\theta,\phi)\sum_{n=0}^{p} v_c^{(p)}(n)\beta_{0,n}\left[\left[\frac{(n+1)\pi}{a}\right]^2+2/r^2-k^2\right]j_0\left(\beta_{0,n}\frac{r}{a}\right), \tag{63}$$

are found from Eqs. (21) and (61) to be given by

$$\mathbf{J}_c^{(p,1,0)}(\mathbf{r})=\mathbf{u}_\phi\frac{c}{4k}\sqrt{\frac{3}{4\pi a^3}}\sin\theta\sum_{n=0}^{p} v_c^{(p)}(n)(n+1)\left[\left[\frac{(n+1)\pi}{a}\right]^2+2/r^2-k^2\right]j_0\left[(n+1)\pi\frac{r}{a}\right], \tag{64}$$

where the coefficients $v_i^{(p)}(n)$ are given by Eqs. (30)–(32). By using Eq. (64) and Eqs. (30)–(32) we obtain

$$\mathbf{J}_c^{(1,1,0)}(\mathbf{r})=\mathbf{u}_\phi\frac{c}{2k}\sqrt{\frac{3}{20\pi a^3}}\sin\theta\left\{\left[\left(\frac{\pi}{a}\right)^2+2/r^2-k^2\right]j_0\left(\pi\frac{r}{a}\right)+\left[\left(\frac{2\pi}{a}\right)^2+2/r^2-k^2\right]j_0\left(2\pi\frac{r}{a}\right)\right\},$$

$$\mathbf{J}_c^{(2,1,0)}(\mathbf{r})=\mathbf{u}_\phi\frac{c}{4k}\sqrt{\frac{3}{280\pi a^3}}\sin\theta\left\{3\left[\left(\frac{\pi}{a}\right)^2+2/r^2-k^2\right]j_0\left(\pi\frac{r}{a}\right)\right.$$
$$\left.-12\left[\left(\frac{2\pi}{a}\right)^2+2/r^2-k^2\right]j_0\left(2\pi\frac{r}{a}\right)-15\left[\left(\frac{3\pi}{a}\right)^2+2/r^2-k^2\right]j_0\left(3\pi\frac{r}{a}\right)\right\},$$

$$\mathbf{J}_c^{(3,1,0)}(\mathbf{r}) = \mathbf{u}_\phi \frac{c}{4k}\sqrt{\frac{1}{140\pi a^3}}\sin\theta\left\{2\left[\left(\frac{\pi}{a}\right)^2 + 2/r^2 - k^2\right]j_0\left(\pi\frac{r}{a}\right) - 8\left[\left(\frac{2\pi}{a}\right)^2 + 2/r^2 - k^2\right]j_0\left(2\pi\frac{r}{a}\right)\right.$$
$$\left. + 18\left[\left(\frac{3\pi}{a}\right)^2 + 2/r^2 - k^2\right]j_0\left(3\pi\frac{r}{a}\right) + 28\left[\left(\frac{4\pi}{a}\right)^2 + 2/r^2 - k^2\right]j_0\left(4\pi\frac{r}{a}\right)\right\}, \qquad (65)$$

$$\mathbf{J}_c^{(4,1,0)}(\mathbf{r}) = \mathbf{u}_\phi \frac{c}{4k}\sqrt{\frac{1}{88\pi a^3}}\sin\theta\left\{\left[\left(\frac{\pi}{a}\right)^2 + 2/r^2 - k^2\right]j_0\left(\pi\frac{r}{a}\right) - 4\left[\left(\frac{2\pi}{a}\right)^2 + 2/r^2 - k^2\right]j_0\left(2\pi\frac{r}{a}\right)\right.$$
$$+ 9\left[\left(\frac{3\pi}{a}\right)^2 + 2/r^2 - k^2\right]j_0\left(3\pi\frac{r}{a}\right) - 16\left[\left(\frac{4\pi}{a}\right)^2 + 2/r^2 - k^2\right]j_0\left(4\pi\frac{r}{a}\right)$$
$$\left. - 30\left[\left(\frac{5\pi}{a}\right)^2 + 2/r^2 - k^2\right]j_0\left(5\pi\frac{r}{a}\right)\right\},$$

$$\mathbf{J}_c^{(5,1,0)}(\mathbf{r}) = \mathbf{u}_\phi \frac{c}{4k}\sqrt{\frac{3}{20020\pi a^3}}\sin\theta\left\{6\left[\left(\frac{\pi}{a}\right)^2 + 2/r^2 - k^2\right]j_0\left(\pi\frac{r}{a}\right)\right.$$
$$- 24\left[\left(\frac{2\pi}{a}\right)^2 + 2/r^2 - k^2\right]j_0\left(2\pi\frac{r}{a}\right) + 54\left[\left(\frac{3\pi}{a}\right)^2 + 2/r^2 - k^2\right]j_0\left(3\pi\frac{r}{a}\right)$$
$$- 96\left[\left(\frac{4\pi}{a}\right)^2 + 2/r^2 - k^2\right]j_0\left(4\pi\frac{r}{a}\right) + 150\left[\left(\frac{5\pi}{a}\right)^2 + 2/r^2 - k^2\right]j_0\left(5\pi\frac{r}{a}\right)$$
$$\left. + 330\left[\left(\frac{6\pi}{a}\right)^2 + 2/r^2 - k^2\right]j_0\left(6\pi\frac{r}{a}\right)\right\},$$

and so on. Finally, it follows from Eq. (50) that $\rho_c^{(p,1,0)}(\mathbf{r}) = 0$ for all $p = 1,2,\ldots$.

We recall[15] that a necessary and sufficient condition for a current distribution $\mathbf{J}_{\mathrm{NR}}(\mathbf{r})$ localized within D to be NR is the vanishing of the multipole moments

$$a_{\ell,m} = -\frac{4\pi}{\ell(\ell+1)}\left(\frac{1}{c}\right)\int_D d^3r\,\mathbf{J}_{\mathrm{NR}}(\mathbf{r})\cdot\{\nabla\times[j_\ell(kr)(\mathbf{L}Y_\ell^m(\theta,\phi))^*]\} = 0,$$
$$b_{\ell,m} = \frac{4\pi i}{\ell(\ell+1)}\left(\frac{k}{c}\right)\int_D d^3r\,\mathbf{J}_{\mathrm{NR}}(\mathbf{r})\cdot[j_\ell(kr)(\mathbf{L}Y_\ell^m(\theta,\phi))^*] = 0. \qquad (66)$$

We show next that—as expected—the basis current distributions with dipolar angular dependence considered here obey the NR conditions (66). By making the substitution $\mathbf{J}_{\mathrm{NR}}(\mathbf{r}) = \mathbf{J}_c^{(p,1,0)}(\mathbf{r})$, with $\mathbf{J}_c^{(p,1,0)}(\mathbf{r})$ given by Eq. (63), into the first of the expressions (66) for the multipole moments while using the second equation of Eq. (16) with $\nu = \ell$ and the fact that the vector functions $\hat{\mathbf{r}}Y_\ell^m(\theta,\phi)$, $\mathbf{L}Y_\ell^m(\theta,\phi)$, and $\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi)$ are mutually perpendicular one concludes that $a_{\ell,m} = 0$. Thus, each of the basis current distributions $\mathbf{J}_c^{(p,1,0)}(\mathbf{r})$ generates zero electric multipole moments $a_{\ell,m}$. This is not surprising since, physically, each of the terms in the series expansion Eq. (63), representing the basis current distributions $\mathbf{J}_c^{(p,1,0)}(\mathbf{r})$, is essentially a radiating magnetic dipole. However, when superposed, the fields produced by these radiating magnetic dipoles cancel out by destructive interference in the region outside the source, i.e., for $r > a$, as we shall show next.

The magnetic multipole moments corresponding to the basis current distribution $\mathbf{J}_c^{(p,1,0)}(\mathbf{r})$ are seen from Eqs. (63) and (66) to be

$$b_{\ell,m}=a^{-3/2}\left[\sum_{n=0}^{p} v_c^{(p)}(n)\beta_{0,n}I(n;k,a)\right]\delta_{\ell,1}\delta_{m,0}, \tag{67}$$

where we have made use of the orthogonality of the vector spherical harmonics $\mathbf{L}Y_\ell^m(\theta,\phi)$ over the unit sphere and where

$$\begin{aligned} I(n;k,a)&=\int_0^a dr\, r^2 j_1(kr)\left[\left[\frac{(n+1)\pi}{a}\right]^2+2/r^2-k^2\right]j_0\left(\beta_{0,n}\frac{r}{a}\right)\\ &=\frac{\pi}{2(k\beta_{0,n}/a)^{1/2}}\left[\left[\frac{(n+1)\pi}{a}\right]^2-k^2\right]\int_0^a dr\, r J_{3/2}(kr)J_{1/2}\left(\beta_{0,n}\frac{r}{a}\right)\\ &\quad+\frac{\pi}{(k\beta_{0,n}/a)^{1/2}}\int_0^a dr(1/r)J_{3/2}(kr)J_{1/2}\left(\beta_{0,n}\frac{r}{a}\right). \end{aligned} \tag{68}$$

It is shown in Appendix D that

$$\int_0^a dr\, u(r)v(r)\left[r(\alpha^2-\kappa^2)-\frac{(n^2-m^2)}{r}\right]=a\left[u(a)\frac{d}{dr}v(r)|_{r=a}-v(a)\frac{d}{dr}u(r)|_{r=a}\right], \tag{69}$$

where

$$\begin{aligned} u(r)&=J_n(\alpha r),\\ v(r)&=J_m(\kappa r), \end{aligned} \tag{70}$$

where n,m,α,κ are arbitrary real numbers. We can now obtain an expression for the integral $I(n;k,a)$ in Eq. (68) by using Eqs. (69) and (70) for $n=3/2$, $m=1/2$, $\alpha=k$, and $\kappa=\beta_{0,n}/a=[(n+1)\pi]/a$. We obtain

$$I(n;k,a)=-\frac{\pi}{2(k\beta_{0,n})^{1/2}}a^{3/2}J_{3/2}(ka)\frac{d}{dr}J_{1/2}\left(\beta_{0,n}\frac{r}{a}\right)\Big|_{r=a}. \tag{71}$$

Now we use $J_{1/2}(x)=\sqrt{(2x/\pi)}j_0(x)$ to express the value of the derivative $(d/dr)\,J_{1/2}(\beta_{0,n}\,r/a)|_{r=a}$ in Eq. (71) in terms of $(d/dr)\,j_0(\beta_{0,n}\,(r/a))|_{r=a}$. We obtain

$$\begin{aligned} I(n;k,a)&=-a^{3/2}\sqrt{\frac{\pi}{2k}}J_{3/2}(ka)\frac{d}{dr}j_0\left(\beta_{0,n}\frac{r}{a}\right)\Big|_{r=a}\\ &=-(-1)^{n+1}\sqrt{\frac{\pi}{2ka}}J_{3/2}(ka). \end{aligned} \tag{72}$$

By substituting from Eq. (72) into expression (67) for the magnetic multipole moments $b_{1,0}$ and imposing the constraint conditions (25) we obtain

$$b_{1,0}=-\left(\frac{\pi^3}{2}\right)^{1/2}k^{-1/2}a^{-2}J_{3/2}(ka)\sum_{n=0}^{p}(-1)^{n+1}(n+1)v_c^{(p)}(n)=0. \tag{73}$$

Finally, it follows from Eqs. (67) and (73) that, as expected, the magnetic multipole moments $b_{\ell,m}=0$ which confirms the NR nature of the basis NR current distributions $\mathbf{J}_c^{(p,1,0)}(\mathbf{r})$.

VI. CONCLUSION

The Devaney–Wolf representation for NR sources was introduced in Ref. 15 and has since played a key role in the inverse source/inverse scattering disciplines. In this paper we have carried out a detailed analysis of this representation in a spherical coordinate system, obtaining new representations and basis functions for NR sources and fields associated with a given spherical domain. We have provided explicit expressions for the NR source representations and basis functions in question. In so doing, we have enhanced their applicability to problems of object reconstruction.

For the sake of clarity and to simplify some of our manipulations, we restricted our attention to continuous NR fields obeying certain continuity and differentiability properties on the boundary $r=a$ of their spherical region of support D (well behaved NR fields). The latter properties were chosen in order to ensure the continuity of the associated NR charge–current distributions on that boundary (such NR sources are therefore compactly supported in D, as desired). The general results developed in the paper can be extended to a broader class of NR sources and fields if one deals with the various vector differential operators in a weak derivative or distributional sense. We plan to use elsewhere some of the results presented here in formulations of inverse source/inverse scattering problems for sources/scatterers obeying prescribed continuity and differentiability constraints (smoothness constraints) in addition to localization constraints. A canonical example is provided by an inverse source problem wherein the unknown source is known *a priori* to be continuous on the boundary $r=a$ of its region of localization D. There the so-called minimum energy solutions[11] would not, in general, represent valid solutions (because of the continuity constraint) and NR source components described by the results derived in this paper would have to be included.

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APPENDIX A: DERIVATION OF EQS. (16) AND (18)

Here we outline our derivation of Eqs. (16) and (18). The first equation of Eq. (16) is obtained via

$$\begin{aligned}\nabla\times\left[j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\hat{\mathbf{r}}Y_\ell^m(\theta,\phi)\right]&=-\hat{\mathbf{r}}\times\nabla\left[j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)Y_\ell^m(\theta,\phi)\right]\\&=-\frac{i}{r}\left\{-i\mathbf{r}\times\nabla\left[j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)Y_\ell^m(\theta,\phi)\right]\right\}\\&=-\frac{i}{r}j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\mathbf{L}Y_\ell^m(\theta,\phi),\end{aligned}$$

where we have used $\mathbf{L}j_\nu(\beta_{\nu,n}r/a)=0$ and $\nabla\times\hat{\mathbf{r}}=0$. The second of Eq. (16) is obtained via

$$\begin{aligned}
\nabla\times\left[j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\mathbf{L}Y_\ell^m(\theta,\phi)\right]
&=\nabla j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\times\mathbf{L}Y_\ell^m(\theta,\phi)+j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\nabla\times\mathbf{L}Y_\ell^m(\theta,\phi)\\
&=\frac{d}{dr}j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi)+ij_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)[-\mathbf{r}\nabla^2Y_\ell^m(\theta,\phi)+\nabla Y_\ell^m(\theta,\phi)]\\
&=\frac{i\ell(\ell+1)}{r}j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\hat{\mathbf{r}}Y_\ell^m(\theta,\phi)+\left[\frac{d}{dr}j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)+\frac{j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)}{r}\right]\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi),
\end{aligned}$$

where we have used Eq. (41) and the operator identities (see Ref. 28, p. 109)

$$\nabla=\hat{\mathbf{r}}\frac{\partial}{\partial r}-\frac{i}{r}\hat{\mathbf{r}}\times\mathbf{L} \tag{A1}$$

and

$$i\nabla\times\mathbf{L}=\mathbf{r}\nabla^2-\nabla\left(1+r\frac{\partial}{\partial r}\right). \tag{A2}$$

The third equation of Eq. (16) is obtained by the manipulations

$$\begin{aligned}
\nabla\times\left[j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi)\right]&=i\nabla\times\left[rj_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\nabla Y_\ell^m(\theta,\phi)\right]\\
&=i\frac{d}{dr}\left[rj_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\right]\hat{\mathbf{r}}\times\nabla Y_\ell^m(\theta,\phi)\\
&=-\frac{1}{r}\frac{d}{dr}\left[rj_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\right][-i\mathbf{r}\times\nabla Y_\ell^m(\theta,\phi)]
\end{aligned}$$

where we have used $\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi)=ir\nabla Y_\ell^m(\theta,\phi)$ (see Ref. 28, p. 109) and $\nabla\times\nabla=0$.

On using Eq. (2.45) in Ref. 28 and

$$\mathbf{L}=-i\mathbf{r}\times\nabla=i\left(\mathbf{u}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}-\mathbf{u}_\phi\frac{\partial}{\partial\theta}\right),$$

where $\mathbf{u}_\theta$ and $\mathbf{u}_\phi$ are the unit vectors in the positive θ and ϕ directions, respectively, we obtain

$$\begin{aligned}
\nabla\cdot\left[j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\hat{\mathbf{r}}Y_\ell^m(\theta,\phi)\right]&=\frac{1}{r^2}\frac{d}{dr}\left[r^2j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\right]Y_\ell^m(\theta,\phi)\\
&=\left[\frac{2}{r}j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)+\frac{d}{dr}j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\right]Y_\ell^m(\theta,\phi)
\end{aligned}$$

and

$$\nabla\cdot\left[j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi)\right]=\frac{i}{r\sin\theta}j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_\ell^m(\theta,\phi).$$

The last expression can be simplified by using (see Ref. 28, p. 109)

$$L^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

so that

$$\nabla\cdot\left[j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi)\right] = -\frac{i}{r}j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)L^2 Y_\ell^m(\theta,\phi)$$
$$= -\frac{i\ell(\ell+1)}{r}j_\nu\left(\beta_{\nu,n}\frac{r}{a}\right)Y_\ell^m(\theta,\phi).$$

The above-mentioned divergence calculations correspond to the first and third equations of Eq. (18). The second equation of Eq. (18) is obtained in a single step using the same procedure.

APPENDIX B: PROCEDURE TO EVALUATE THE COEFFICIENTS $v_j^{(p)}(n)$

It follows from Eq. (25) that

$$v_j^{(1)}(1) = \tfrac{1}{2}v_j^{(1)}(0). \tag{B1}$$

On the other hand, Eq. (27) with $p=p'=1$ yields

$$|v_j^{(1)}(0)|^2 + |v_j^{(1)}(1)|^2 = 1. \tag{B2}$$

Without loss of generality we choose $v_j^{(p)}(0)$ to be a real and positive coefficient. Then we obtain from Eqs. (B1) and (B2)

$$v_j^{(1)}(0) = 2/\sqrt{5},$$
$$v_j^{(1)}(1) = 1/\sqrt{5}. \tag{B3}$$

Equation (27) with $p=1$ and $p'=2$ yields

$$v_j^{(2)}(1) = -2v_j^{(2)}(0). \tag{B4}$$

Equation (27) with $p=p'=2$ yields

$$|v_j^{(2)}(0)|^2 + |v_j^{(2)}(1)|^2 + |v_j^{(2)}(2)|^2 = 1. \tag{B5}$$

It follows from Eq. (25) that

$$-v_j^{(2)}(0) + 2v_j^{(2)}(1) - 3v_j^{(2)}(2) = 0. \tag{B6}$$

By solving simultaneously Eqs. (B4), (B5), and (B6) while requiring $v_j^{(2)}(0)$ to be real and positive we obtain

$$v_j^{(2)}(0) = 3/\sqrt{70},$$
$$v_j^{(2)}(1) = -6/\sqrt{70}, \tag{B7}$$
$$v_j^{(2)}(2) = -5/\sqrt{70}.$$

APPENDIX C: DERIVATION OF EQ. (42)

The first equation of Eq. (42) is obtained from the first equation of Eq. (16), with $\nu=0$, and Eqs. (41), (A1), (A2) via

$$\nabla\times\nabla\times\left[j_0\left(\beta_{0,n}\frac{r}{a}\right)\hat{\mathbf{r}}Y_\ell^m(\theta,\phi)\right]$$

$$=-i\nabla\times\left[\frac{1}{r}j_0\left(\beta_{0,n}\frac{r}{a}\right)\mathbf{L}Y_\ell^m(\theta,\phi)\right]$$

$$=-i\nabla\left[\frac{1}{r}j_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\times\mathbf{L}Y_\ell^m(\theta,\phi)-\frac{i}{r}j_0\left(\beta_{0,n}\frac{r}{a}\right)\nabla\times\mathbf{L}Y_\ell^m(\theta,\phi)$$

$$=-i\frac{d}{dr}\left[\frac{1}{r}j_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi)-\frac{1}{r}j_0\left(\beta_{0,n}\frac{r}{a}\right)(\mathbf{r}\nabla^2-\nabla)Y_\ell^m(\theta,\phi)$$

$$=-i\left\{\frac{d}{dr}\left[\frac{1}{r}j_0\left(\beta_{0,n}\frac{r}{a}\right)\right]+\frac{1}{r^2}j_0\left(\beta_{0,n}\frac{r}{a}\right)\right\}\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi)+\frac{\ell(\ell+1)}{r^2}j_0\left(\beta_{0,n}\frac{r}{a}\right)\hat{\mathbf{r}}Y_\ell^m(\theta,\phi)$$

$$=\frac{\ell(\ell+1)}{r^2}j_0\left(\beta_{0,n}\frac{r}{a}\right)\hat{\mathbf{r}}Y_\ell^m(\theta,\phi)-\frac{i}{r}\frac{d}{dr}j_0\left(\beta_{0,n}\frac{r}{a}\right)\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi).$$

The second equation of Eq. (42) follows from Eq. (16), with $\nu=0$, and Eq. (40) via

$$\nabla\times\nabla\times\left[j_0\left(\beta_{0,n}\frac{r}{a}\right)\mathbf{L}Y_\ell^m(\theta,\phi)\right]$$

$$=\nabla\times\left\{\frac{i\ell(\ell+1)}{r}j_0\left(\beta_{0,n}\frac{r}{a}\right)\hat{\mathbf{r}}Y_\ell^m(\theta,\phi)\right.$$

$$\left.+\frac{1}{r}\frac{d}{dr}\left[rj_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi)\right\}$$

$$=-\frac{i}{r}\left[\frac{i\ell(\ell+1)}{r}j_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\mathbf{L}Y_\ell^m(\theta,\phi)-\frac{1}{r}\frac{d}{dr}\left\{r\left[\frac{1}{r}\frac{d}{dr}\left[rj_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\right]\right\}\mathbf{L}Y_\ell^m(\theta,\phi)$$

$$=\frac{\ell(\ell+1)}{r^2}j_0\left(\beta_{0,n}\frac{r}{a}\right)\mathbf{L}Y_\ell^m(\theta,\phi)-\frac{1}{r}\frac{d^2}{dr^2}\left[rj_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\mathbf{L}Y_\ell^m(\theta,\phi)$$

$$=\left\{\left[\frac{\ell(\ell+1)}{r^2}-\nabla^2\right]j_0\left(\beta_{0,n}\frac{r}{a}\right)\right\}\mathbf{L}Y_\ell^m(\theta,\phi)$$

$$=\left\{\left[\frac{(n+1)\pi}{a}\right]^2+\frac{\ell(\ell+1)}{r^2}\right\}j_0\left(\beta_{0,n}\frac{r}{a}\right)\mathbf{L}Y_\ell^m(\theta,\phi).$$

The third equation of Eq. (42) follows from Eq. (16), with $\nu=0$, and Eq. (40) via

$$\nabla\times\nabla\times\left[j_0\left(\beta_{0,n}\frac{r}{a}\right)\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi)\right]$$

$$=-\nabla\times\left\{\frac{1}{r}\frac{d}{dr}\left[rj_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\mathbf{L}Y_\ell^m(\theta,\phi)\right\}$$

$$=-\frac{i\ell(\ell+1)}{r}\left\{\frac{1}{r}\frac{d}{dr}\left[rj_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\right\}\hat{\mathbf{r}}Y_\ell^m(\theta,\phi)-\frac{1}{r}\frac{d}{dr}\left\{r\left[\frac{1}{r}\frac{d}{dr}\left[rj_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\right]\right\}\hat{\mathbf{r}}$$

$$\times\mathbf{L}Y_\ell^m(\theta,\phi)$$

$$= -\frac{i\ell(\ell+1)}{r^2}\frac{d}{dr}\left[r j_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\hat{\mathbf{r}}Y_\ell^m(\theta,\phi) - \nabla^2 j_0\left(\beta_{0,n}\frac{r}{a}\right)\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi)$$

$$= -\frac{i\ell(\ell+1)}{r^2}\frac{d}{dr}\left[r j_0\left(\beta_{0,n}\frac{r}{a}\right)\right]\hat{\mathbf{r}}Y_\ell^m(\theta,\phi) + \left[\frac{(n+1)\pi}{a}\right]^2 j_0\left(\beta_{0,n}\frac{r}{a}\right)\hat{\mathbf{r}}\times\mathbf{L}Y_\ell^m(\theta,\phi).$$

APPENDIX D: EVALUATION OF THE INTEGRAL IN EQ. (69)

Our starting point is the Bessel equations

$$\begin{aligned} x u''(x) + u'(x) + x(\alpha^2 - n^2/x^2)u(x) = 0,\\ x v''(x) + v'(x) + x(\kappa^2 - m^2/x^2)v(x) = 0, \end{aligned} \tag{D1}$$

where

$$\begin{aligned} u(x) = J_n(\alpha x),\\ v(x) = J_m(\kappa x), \end{aligned} \tag{D2}$$

where n, m, α, κ are arbitrary real numbers and the primes ($'$) are used to denote derivatives with respect to x. By multiplying the first equation of Eq. (D1) by $v(x)$ and the second equation of Eq. (D1) by $u(x)$ we obtain

$$x v(x) u''(x) + u'(x) v(x) + x(\alpha^2 - n^2/x^2)u(x)v(x) = 0 \tag{D3}$$

and

$$x v''(x) u(x) + v'(x) u(x) + x(\kappa^2 - m^2/x^2)u(x)v(x) = 0. \tag{D4}$$

By integrating both sides of Eq. (D3) and (D4) from $x=0$ to $x=a$ and subtracting the resulting equations one obtains after some manipulations the desired result Eq. (69). The procedure is similar to that used in Ref. 28, pp. 591–592, to derive the orthogonality relation for the Bessel functions.

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