Inverse source problem and minimum-energy sources

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We present a new linear inversion formalism for the scalar inverse source problem in three-dimensional and one-dimensional (1D) spaces, from which a number of previously unknown results on minimum-energy (ME) sources and their fields readily follow. ME sources, of specified support, are shown to obey a homogeneous Helmholtz equation in the interior of that support. As a consequence of that result, the fields produced by ME sources are shown to obey an iterated homogeneous Helmholtz equation. By solving the latter equation, we arrive at a new Green-function representation of the field produced by a ME source. It is also shown that any square-integrable (L^2), compactly supported source that possesses a continuous normal derivative on the boundary of its support must possess a nonradiating (NR) component. A procedure based on our results on the inverse source problem and ME sources is described to uniquely decompose an L^2 source of specified support and its field into the sum of a radiating and a NR part. The general theory that is developed is illustrated for the special cases of a homogeneous source in 1D space and a spherically symmetric source. [S0740-3232(00)01901-3]

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1. INTRODUCTION

In three-dimensional (3D) space $\Re^3$, the inverse source problem to the inhomogeneous Helmholtz equation

$$(\nabla^2 + k^2)\psi(\mathbf{r}) = -4\pi\rho(\mathbf{r}) \qquad (k \neq 0), \tag{1}$$

where $\rho(\mathbf{r})$ is a source of known support D, can be stated as being that of deducing $\rho(\mathbf{r})$ from knowledge of the radiated field

$$\psi(\mathbf{r}) = \int_D \mathrm{d}^3 r' \, \rho(\mathbf{r}') \frac{\exp(ik|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} \tag{2}$$

at all observation points $\mathbf{r} \notin D$ not contained in the source region D. Henceforth we shall assume D to be a volume contained within the spherical volume $\tau = \{\mathbf{r}: \mathbf{r} \in \Re^3, r \leqslant a\}$ of radius $a > 0$ and center at the origin $\mathbf{r} = 0$, i.e., $D \subseteq \tau$.

The inverse source problem does not admit a unique solution because of the possible existence of nonradiating (NR) sources[1,2] within the source's support D. The latter generate fields that vanish for $\mathbf{r} \notin D$. Therefore, if $\hat{\rho}(\mathbf{r})$ is a solution to the inverse source problem, then $\hat{\rho}'(\mathbf{r}) = \hat{\rho}(\mathbf{r}) + \rho_{\mathrm{NR}}(\mathbf{r})$, where $\rho_{\mathrm{NR}}(\mathbf{r})$ is a NR source localized within D, is also a solution. It is well-known[3] that the inverse source problem admits a unique solution if we require it to possess a minimum L^2 norm

$$\left[\int_D \mathrm{d}^3 r' |\hat{\rho}(\mathbf{r}')|^2\right]^{1/2}$$

among all solutions $\hat{\rho}(\mathbf{r})$. This particular solution, henceforth to be denoted as $\rho_{\mathrm{ME}}(\mathbf{r})$, is usually termed the minimum-energy (ME) solution.[3] We shall refer sometimes to ME solutions simply as ME sources.

Although ME sources have appeared before in treatments of the inverse source problem,[3,4] relatively little is known about their properties and those of the fields that they produce. Recently, two of us (EAM and AJD) presented a new treatment of the electromagnetic inverse source problem based on a linear inversion formalism in Hilbert spaces and multipole expansions.[5] In this paper we report the scalar counterpart of that analysis in 3D and one-dimensional (1D) spaces, from which a number of previously unknown results on ME sources and their fields readily follow. Unlike that of previous workers in this area,[3,4] our focus is on both the radiation and source-reconstruction aspects of these sources and their fields.

In Section 2 we develop a linear inversion formalism for the scalar inverse source problem in 3D space, valid for square-integrable (L^2) sources of specified support D. Our analysis makes use of standard linear inversion and multipole theories. The ME solution to the inverse source problem is given explicitly for the special case where $D = \tau$ (a source whose known support is the spherical volume τ). Among other results, ME sources

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$\rho_{\mathrm{ME}}(\mathbf{r})$ of support D are found to obey the homogeneous Helmholtz equation (see Theorem 1)

$$(\nabla^2 + k^2)\rho_{\mathrm{ME}}(\mathbf{r}) = 0 \tag{3}$$

in the interior of the volume D, excluding its boundary (which we shall denote as ∂D). Using Eq. (3) and standard results of linear inversion theory, we show that any (nontrivial) L^2, compactly supported source that possesses a continuous normal derivative on the boundary of its support must possess a NR part (Theorem 3). This previously unknown result speaks about an intrinsically unobservable component associated with a physically interesting class of sources. It also follows from Eq. (3) that the fields produced by ME sources of support D satisfy the iterated homogeneous Helmholtz equation (see Theorem 2)

$$(\nabla^2 + k^2)^2\psi(\mathbf{r}) = 0 \tag{4}$$

everywhere except on the boundary ∂D of D. Theorems 1–3 are, to the best of our knowledge, new. The theory leading to Theorem 1 has been available since the early work of Bleistein and Cohen[2] on the inverse source problem and appears to have been overlooked by other workers in this field.[3,4] In Section 3 we solve Eq. (4), obtaining a new Green-function expansion for the field radiated by a ME source. Unlike the outgoing Green-function integral, the new expansion is given in terms of the value of the source and its normal derivative on the surface ∂D that bounds the volume D. In Section 4 we present the 1D counterpart of the source-inversion formulation developed in Section 2. We also show how some of our results in Section 2 can be used to uniquely decompose a known source and its field into the sum of a radiating and a NR part. The special cases of a homogeneous 1D source and a homogeneous spherical source of a given size are examined in detail. Our work in this regard adds to the recent work of Berry *et al.*[6] on nonpropagating string excitations and to that of Kim and Wolf[7] on NR homogeneous spherical sources. Our results apply to both radiating and NR homogeneous sources and coincide with those given in Refs. 6 and 7 when the homogeneous source is NR. Section 5 provides a summary of the main results derived in the paper.

2. INVERSE SOURCE PROBLEM AND MINIMUM-ENERGY SOURCES

In the following, attention is restricted to sources $\rho \in X$ in the Hilbert space X of L^2 functions of $\mathbf{r} \in \mathfrak{R}^3$ localized within D, to which we assign the inner product

$$(\rho, \rho')_X = \int_D \mathrm{d}^3 r' \, \rho^*(\mathbf{r}')\rho'(\mathbf{r}'), \tag{5}$$

where $*$ denotes the complex conjugate.

A. Linear Inversion Formulation

It is well-known[8] that for $r > a$ the field $\psi(\mathbf{r})$ radiated by a source $\rho(\mathbf{r})$ confined within $D \subseteq \tau$ can be expressed as the multipole expansion

$$\psi(\mathbf{r}) = ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_{l,m} h_l^{(1)}(kr) Y_{l,m}(\hat{\mathbf{r}}), \tag{6}$$

where $\hat{\mathbf{r}} \equiv \mathbf{r}/r$, $h_l^{(1)}(\,\cdot\,)$ is the spherical Hankel function of the first kind and order l (as defined in Ref. 8, p. 740), and $Y_{l,m}(\,\cdot\,)$ is the spherical harmonic of degree l and order m (as defined in Ref. 8, p. 99). The expansion coefficients $g_{l,m}$ in Eq. (6) are the multipole moments and are defined by the inner products

$$g_{l,m} = (\psi_{l,m}, \rho)_X, \tag{7}$$

where

$$\psi_{l,m}(\mathbf{r}) = 4\pi M(\mathbf{r}) j_l(kr) Y_{l,m}(\hat{\mathbf{r}}), \qquad l = 0, 1, \ldots; \quad m = -l, -l+1, \ldots, l, \tag{8}$$

where $j_l(\,\cdot\,)$ is the spherical Bessel function of the first kind and order l (as defined in Ref. 8, p. 740) and

$$M(\mathbf{r}) = \begin{cases} 1 & \text{if } \mathbf{r} \in D \\ 0 & \text{else} \end{cases}. \tag{9}$$

The inverse source problem can be addressed in a framework for linear inverse problems developed by Bertero and co-workers.[9] In this framework the multipole moments $g_{l,m}$ can be viewed as the entries of a data vector $\mathbf{g} = \{g_{l,m}\}$. We let Y be the data space of square-summable vectors $\mathbf{g}$, where $\Sigma_{l=0}^{\infty}\Sigma_{m=-l}^{l}|g_{l,m}|^2 < \infty$, and assign to it the inner product

$$(\mathbf{g}, \mathbf{g}')_Y = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_{l,m}^* g'_{l,m}. \tag{10}$$

The inverse source problem can be formulated as being that of deducing $\rho \in X$ from knowledge of $\mathbf{g} \in Y$.

We can write the forward relation Eq. (7) as

$$L\rho = \mathbf{g}, \tag{11}$$

where $L: X \to Y$ is a linear mapping that transforms a function in X [i.e., the source $\rho(\mathbf{r})$] into a vector of Y (i.e., the data vector $\mathbf{g}$) according to the rule

$$(L\rho)_{l,m} = (\psi_{l,m}, \rho)_X. \tag{12}$$

A source $\rho_{\mathrm{NR}} \in X$ is NR if and only if $(\psi_{l,m}, \rho_{\mathrm{NR}})_X = 0$ for all $l = 0,1,\ldots;\ m = -l, -l+1, \ldots, l$.[1] Thus the class of L^2 NR sources of support D is exactly the null space $N(L) = \{\rho \in X: L\rho = 0\}$ of L.

The adjoint $L^\dagger$ of L, defined by

$$(L\rho, \mathbf{g})_Y = (\rho, L^\dagger \mathbf{g})_X, \tag{13}$$

is found from Eqs. (5), (10), and (12) to be given by

$$(L^\dagger \mathbf{g})(\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_{l,m}\psi_{l,m}(\mathbf{r}). \tag{14}$$

We see from Eqs. (8), (12), and (14) that $L^\dagger$ maps $Y \to X$, while the operators $LL^\dagger$ and $L^\dagger L$ map $Y \to Y$ and $X \to X$, respectively.

It will be assumed throughout that the data vector $\mathbf{g}$ is noise free, i.e., $\mathbf{g} \in R(L) = \{\mathbf{g} \in Y: \mathbf{g} = L\rho, \rho \in X\}$, where $R(L)$ is the range of L. This not only ensures the existence of exact (although nonunique) solutions to the inverse source problem but also enables us to focus on the intrinsic properties of ME sources and their fields rather than on the computational issues of practical inverse source problems.[10] A unique solution to the inverse source problem cannot be obtained in general because of the possible existence of NR sources $\rho_{\mathrm{NR}}(\mathbf{r}) \in N(L)$. Uniqueness can be enforced, however, by imposing the additional constraint of minimizing the source's L^2 norm. The solution in question is the usual ME solution, also known as the normal solution in linear inversion language.[9] Physical interpretations of these ME solutions have been given in papers by Devaney and Porter[3,4] and Langenberg[11] in the context of generalized holography.

The ME solution corresponding to a given data vector $\mathbf{g} \in R(L)$ can be evaluated by using the pseudoinverse of L, i.e.,[12]

$$\rho_{\mathrm{ME}} = L^{\dagger}\tilde{\mathbf{g}}, \tag{15}$$

where

$$\tilde{\mathbf{g}} = (LL^{\dagger})^{-1}\mathbf{g} \tag{16}$$

is a filtered data vector associated with the given data vector $\mathbf{g}$. We show in Subsection 2.A.1 that for a source whose support D is the spherical volume τ the first step [Eq. (16)] of the two-step procedure [Eqs. (15) and (16)] can be regarded as a filtering step that associates with a given data vector $\mathbf{g}$ the multipole moments $\tilde{g}_{l,m} = g_{l,m}/\sigma_l^2$ (the filtered data), where σ_l are the singular values of the linear mapping L. The second step [Eq. (15)], with $L^{\dagger}$ defined by Eq. (14), can be regarded as a backpropagation into the source region of the corresponding filtered data field. Equation (16) thus generates a filtered data vector associated with a filtered data field, which, once backpropagated according to Eqs. (14) and (15), yields the corresponding ME source.

1. Special Case: Source in Spherical Region

It is not hard to show by using Eqs. (8), (9), (12), and (14)–(16) and a procedure analogous to that used in Ref. 5 for the electromagnetic case that, for a source whose support D is the spherical volume τ,

$$\begin{aligned}\rho_{\mathrm{ME}}(\mathbf{r}) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_{l,m}\psi_{l,m}(\mathbf{r})/\sigma_l^2 \\ &= 4\pi M(\mathbf{r})\sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_{l,m}j_l(kr)Y_{l,m}(\hat{\mathbf{r}})/\sigma_l^2,\end{aligned} \tag{17}$$

where

$$\begin{aligned}\sigma_l^2 &= (4\pi)^2\int_0^a \mathrm{d}r'\, r'^2 j_l^2(kr') \\ &= 8\pi^2a^3[j_l^2(ka) - j_{l-1}(ka)j_{l+1}(ka)].\end{aligned} \tag{18}$$

The ME source is thus given by a source-free multipole expansion with multipole moments $\tilde{g}_{l,m} = g_{l,m}/\sigma_l^2$ truncated within the source's support. The terms σ_l^2 are known to decay exponentially fast for $l > ka$, confirming the ill-posed nature of the inverse source problem.[12] Equations (17) and (18) have been derived in Ref. 12 by means of the singular-value decomposition technique in connection with the inverse source problem with far-field data and in Refs. 3 and 4 in the context of the Porter–Bojarski integral equation.

2. Class of Minimum-Energy Sources

It follows from Eq. (15) that $\rho_{\mathrm{ME}} \in R(L^{\dagger}) = \{\rho \in X: \rho = L^{\dagger}\mathbf{g}, \mathbf{g} \in Y\}$. Furthermore, by noting that[12]

$$[N(L)]^{\perp} = \overline{R(L^{\dagger})}, \tag{19}$$

where $[N(L)]^{\perp} \in X$ is the orthogonal complement of $N(L)$ and $\overline{R(L^{\dagger})} \subset X$ is the closure of $R(L^{\dagger})$, one concludes that $\rho_{\mathrm{ME}} \in [N(L)]^{\perp}$. Now it follows from Eq. (19) and the projection theorem[13] that any source $\rho \in X$ can be uniquely decomposed into the sum of a radiating and a NR part, $\rho_1 \in [N(L)]^{\perp}$ and $\rho_{\mathrm{NR}} \in N(L)$, respectively, i.e.,

$$X = N(L) \oplus [N(L)]^{\perp}. \tag{20}$$

The orthogonal complement $[N(L)]^{\perp}$ of $N(L)$ can be shown to be exactly the space of ME sources characterized above. By this we mean that any $\rho_1 \in N(L)^{\perp}$ is a ME source in the Hilbert space X and, vice versa, any ME source in the Hilbert space X is also a member of $[N(L)]^{\perp}$. The second condition was established above in connection with Eq. (19). It remains to be shown that all sources $\rho_1 \in [N(L)]^{\perp}$ are ME sources. To show this, we note that the most general solution to the inverse source problem for a data vector $\mathbf{g} = L\rho_1 \in Y$ must be expressible as $\hat{\rho}(\mathbf{r}) = \rho_1(\mathbf{r}) + \rho_{\mathrm{NR}}(\mathbf{r})$, where $\rho_{\mathrm{NR}} \in N(L)$. By using this observation and the orthogonality of the subspaces $N(L)$ and $[N(L)]^{\perp}$, one verifies that $(\rho_1, \rho_1)_X \leqslant (\hat{\rho}, \hat{\rho})_X$, which establishes $\rho_1 \in [N(L)]^{\perp}$ as the corresponding ME solution. The orthogonal complement $[N(L)]^{\perp}$ of $N(L)$ is thus exactly the space of ME sources associated with the Hilbert space X. The properties of ME sources and their fields, to be discussed next, are therefore of interest for both direct and inverse source problems.

B. Theorems: Wave Properties of Minimum-Energy Sources

It follows from Eqs. (14) and (15) that the ME source $\rho_{\mathrm{ME}}(\mathbf{r})$ is given by a series expansion over the functions $\psi_{l,m}(\mathbf{r})$, each of which is truncated within the source's support D and obeys a homogeneous Helmholtz equation, i.e.,

$$(\nabla^2 + k^2)\psi_{l,m}(\mathbf{r}) = 0$$

in the interior of the domain D, its boundary ∂D excluded. In addition, it can be shown by using the Picard conditions[12] that define the range $R(L)$ of L and a theorem that is due to Müller (see Ref. 14, p. 72–73) that the series expansion defined by Eqs. (14)–(16), corresponding to the ME source $\rho_{\mathrm{ME}}(\mathbf{r})$, must converge uniformly and absolutely in the interior of the region D if $\mathbf{g} \in R(L)$. For a source contained in the spherical volume $D = \tau$, the latter requirement translates into

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} |g_{l,m}|^2/\sigma_l^2 < \infty, \tag{21}$$

which is seen to ensure that the ME solution in Eqs. (17) and (18) is L^2. The following theorem follows at once from these observations, the results of Subsection 2.A.2, and the differentiability properties that apply to any uniformly convergent series (see Ref. 15, p. 302).

Theorem 1. Let $\rho_{\rm ME} \in [N(L)]^{\perp}$. Then $\rho_{\rm ME}(\mathbf{r})$ obeys the homogeneous Helmholtz equation (3) in the interior of the volume D, its boundary ∂D excluded.

Since only the radiating part $\rho_1 \in [N(L)]^{\perp}$ of a source $\rho \in X$ contributes to its field for $\mathbf{r} \notin D$, one concludes that the primary source of wave radiation is, by itself, a wave truncated within the source's support. The following theorem follows at once from Theorem 1.

Theorem 2. The field $\psi(\mathbf{r})$ generated by a ME source $\rho_{\rm ME} \in X$ obeys the iterated homogeneous Helmholtz equation (4) everywhere except on the boundary ∂D of the volume D.

Proof. That Eq. (4) is valid for $\mathbf{r} \notin D$ is trivial, since $(\nabla^2 + k^2)\psi(\mathbf{r}, \omega) = 0$ for $\mathbf{r} \notin D$, as follows from Eq. (1). That Eq. (4) is valid in the interior of the volume D follows by applying the $\nabla^2 + k^2$ operator to both sides of Eq. (1) with $\rho(\mathbf{r}) = \rho_{\rm ME}(\mathbf{r})$ and using Eq. (3).

Theorems 1 and 2 are the basis of the Green-function representations of ME sources and their fields, to be discussed in Section 3. Here we wish to address another question: Can a source $\rho_{\rm ME} \in [N(L)]^{\perp}$ of compact support D possess a continuous normal derivative on the boundary ∂D of D? This is not possible, since any such source $\rho_{\rm ME}(\mathbf{r})$ would necessarily obey, in view of Theorem 1, the homogeneous Helmholtz equation (3) inside the volume D, subject to the overspecified boundary conditions $\rho_{\rm ME}(\mathbf{r}) = 0$ and $(\partial/\partial n)\rho_{\rm ME}(\mathbf{r}) = 0$ on ∂D (where $\partial/\partial n$ is the partial derivative with respect to the outward-directed normal to ∂D). Then, as is well known (see Ref. 16, Chap. 7), $\rho_{\rm ME}(\mathbf{r}) = 0$. This result and our discussion in Subsection 2.A.2 lead at once to one of the central results of this paper:

Theorem 3. Let $\rho \in X$ possess compact support D and a continuous normal derivative on the boundary ∂D of D. Then $\rho(\mathbf{r})$ must possess a NR part.

The source restrictions imposed in Theorem 3 can be relaxed. In particular, there are more general classes of sources other than the ones in Theorem 3 that do not obey Eq. (3) and therefore also possess a NR part. For example, all L^2 sources of compact support in D (not necessarily differentiable on the boundary ∂D of D), excluding resonant wave solutions of Eq. (3) that vanish on ∂D, must possess a NR component.

3. GREEN-FUNCTION REPRESENTATIONS OF MINIMUM-ENERGY SOURCES AND THEIR FIELDS

We next derive new Green-function representations of ME sources of arbitrary support and their fields.

A. Green-Function Representation of Minimum-Energy Sources

A Green-function representation of ME sources is obtained by solving Eq. (3) with the aid of a Green function suited to prescribed Dirichlet and/or Newmann conditions on the boundary ∂D of the volume D. Let $G(\mathbf{r}, \mathbf{r}')$ be a Green function of the Helmholtz operator, so that

$$(\nabla^2 + k^2)G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}'). \tag{22}$$

On using Eqs. (3) and (22) and the Green function $G_D(\mathbf{r}, \mathbf{r}')$ that satisfies homogeneous Dirichlet conditions on ∂D (corresponding to the interior problem), one obtains, by means of the usual procedure,[16]

$$-\frac{1}{4\pi}\int_{\partial D} \mathrm{d}S'\, \rho_{\rm ME}(\mathbf{r}')\frac{\partial}{\partial n'}G_D(\mathbf{r}, \mathbf{r}') = \begin{cases} \rho_{\rm ME}(\mathbf{r}) & \text{if } \mathbf{r} \in D \\ 0 & \text{if } \mathbf{r} \notin D \end{cases}, \tag{23}$$

where the bottom equation is a statement of the well-known extinction theorem. In Eq. (23), $\partial/\partial n'$ is the partial derivative with respect to the outward-directed normal to the surface ∂D. On using the free-space Green function $G_0(\mathbf{r}, \mathbf{r}') = \exp(ik|\mathbf{r} - \mathbf{r}'|)/|\mathbf{r} - \mathbf{r}'|$, one obtains the Helmholtz–Kirchhoff integral[17]

$$\frac{1}{4\pi}\int_{\partial D} \mathrm{d}S' \left[G_0(\mathbf{r}, \mathbf{r}')\frac{\partial}{\partial n'}\rho_{\rm ME}(\mathbf{r}') - \rho_{\rm ME}(\mathbf{r}')\frac{\partial}{\partial n'}G_0(\mathbf{r}, \mathbf{r}')\right] = \begin{cases} \rho_{\rm ME}(\mathbf{r}) & \text{if } \mathbf{r} \in D \\ 0 & \text{if } \mathbf{r} \notin D \end{cases}. \tag{24}$$

B. Green-Function Representation of the Field Generated by a Minimum-Energy Source

The field $\psi(\mathbf{r})$ radiated by the ME source $\rho_{\rm ME}(\mathbf{r})$ is given by Eq. (2) with $\rho(\mathbf{r}) = \rho_{\rm ME}(\mathbf{r})$. Although Eq. (2) with $\rho(\mathbf{r}) = \rho_{\rm ME}(\mathbf{r})$ is certainly a solution, it involves $\rho_{\rm ME}(\mathbf{r})$ throughout the interior of D and is thus overspecified. What is required is a solution in terms of the Dirichlet and Newmann conditions of $\rho_{\rm ME}(\mathbf{r})$ on ∂D only [see Eq. (24)]. In principle, one should be able to obtain a solution in terms of one of the above conditions, i.e., $\rho_{\rm ME}(\mathbf{r})$ or $(\partial/\partial n)\rho_{\rm ME}(\mathbf{r})$ on ∂D [see Eq. (23)]. Two approaches are presented below. One makes use of results derived in Ref. 18 (see Subsection 3.B.2 below); the other is based on direct substitution of Eq. (24) into Eq. (2) (see Subsection 3.B.3).

1. Representation of the Field Outside the Source Volume

Before an expression is obtained for the radiated field in terms of the Dirichlet and Newmann conditions of $\rho_{\rm ME}(\mathbf{r})$ on ∂D, it is useful to give the solution in the region outside D in terms of the Dirichlet and Newmann conditions of the field on ∂D. This solution is obtained in the usual way by using Eqs. (1) and (22) and the free-space Green function $G_0(\mathbf{r}, \mathbf{r}')$ (Ref. 16):

$$\int_D \mathrm{d}^3 r' \, \rho_{\mathrm{ME}}(\mathbf{r}') G_0(\mathbf{r}, \mathbf{r}') + \frac{1}{4\pi} \int_{\partial D} \mathrm{d}S' \left[G_0(\mathbf{r}, \mathbf{r}') \frac{\partial}{\partial n'} \psi(\mathbf{r}') - \psi(\mathbf{r}') \frac{\partial}{\partial n'} G_0(\mathbf{r}, \mathbf{r}') \right. = \begin{cases} \psi(\mathbf{r}) & \text{if } \mathbf{r} \in D \\ 0 & \text{if } \mathbf{r} \notin D \end{cases}. \quad (25)$$

In view of Eqs. (2) and (25), it follows at once that (see also Ref. 11)

$$\frac{1}{4\pi} \int_{\partial D} \mathrm{d}S' \left[G_0(\mathbf{r}, \mathbf{r}') \frac{\partial}{\partial n'} \psi(\mathbf{r}') - \psi(\mathbf{r}') \frac{\partial}{\partial n'} G_0(\mathbf{r}, \mathbf{r}') \right] = \begin{cases} -\psi(\mathbf{r}) & \text{if } \mathbf{r} \notin D \\ 0 & \text{if } \mathbf{r} \in D \end{cases}. \quad (26)$$

2. *General Green-Function Solution*

It is shown in Appendix A that the solution of Eq. (4) can be expressed as

$$\psi(\mathbf{r}) = \frac{1}{8\pi k} \int_{\partial D} \mathrm{d}S' \left[(\nabla'^2 + k^2)\psi(\mathbf{r}') \frac{\partial^2}{\partial k \partial n'} G_0(\mathbf{r}, \mathbf{r}') - \frac{\partial}{\partial n'} (\nabla'^2 + k^2)\psi(\mathbf{r}') \frac{\partial}{\partial k} G_0(\mathbf{r}, \mathbf{r}') \right] + \frac{1}{4\pi} \int_{\partial D} \mathrm{d}S' \left[G_0(\mathbf{r}, \mathbf{r}') \frac{\partial}{\partial n'} \psi(\mathbf{r}') - \psi(\mathbf{r}') \frac{\partial}{\partial n'} G_0(\mathbf{r}, \mathbf{r}') \right]. \quad (27)$$

On using Eq. (26), we note that the second integral in Eq. (27) vanishes inside D. Furthermore, we note from Eq. (1) that the boundary value $(\nabla'^2 + k^2)\psi(\mathbf{r}')$ in Eq. (27) can be replaced with $-4\pi\rho_{\mathrm{ME}}(\mathbf{r}')$, giving

$$\psi(\mathbf{r}) = \frac{1}{2k} \int_{\partial D} \mathrm{d}S' \left[\frac{\partial}{\partial n'} \rho_{\mathrm{ME}}(\mathbf{r}') \frac{\partial}{\partial k} G_0(\mathbf{r}, \mathbf{r}') - \rho_{\mathrm{ME}}(\mathbf{r}') \frac{\partial^2}{\partial k \partial n'} G_0(\mathbf{r}, \mathbf{r}') \right], \quad (28)$$

which is the sought-after Green-function representation of $\psi(\mathbf{r})$ in terms of the Dirichlet and Newmann conditions of $\rho_{\mathrm{ME}}(\mathbf{r})$ on ∂D, valid throughout the source volume D. We can extend the domain over which Eq. (28) holds by noting from the extinction theorem for the iterated Helmholtz equation (4) that, for $\mathbf{r} \notin D$,

$$\frac{1}{8\pi k} \int_{\partial D} \mathrm{d}S' \left[(\nabla'^2 + k^2)\psi(\mathbf{r}') \frac{\partial^2}{\partial k \partial n'} G_0(\mathbf{r}, \mathbf{r}') - \frac{\partial}{\partial n'} (\nabla'^2 + k^2)\psi(\mathbf{r}') \frac{\partial}{\partial k} G_0(\mathbf{r}, \mathbf{r}') \right. = \frac{1}{4\pi} \int_{\partial D} \mathrm{d}S' \left[\psi(\mathbf{r}') \frac{\partial}{\partial n'} G_0(\mathbf{r}, \mathbf{r}') - G_0(\mathbf{r}, \mathbf{r}') \frac{\partial}{\partial n'} \psi(\mathbf{r}') \right], \quad (29)$$

which, on using Eq. (26) [while replacing $(\nabla'^2 + k^2)\psi(\mathbf{r}')$ with $-4\pi\rho_{\mathrm{ME}}(\mathbf{r}')$], shows that Eq. (28) actually holds both inside and outside D. Interestingly, for $\mathbf{r} \notin D$, Eq. (28) is seen to have the same form as that of Eq. (26) if $\rho_{\mathrm{ME}}(\mathbf{r})(2\pi/k)(\partial/\partial k)$ is replaced with $-\psi(\mathbf{r})$.

3. *Direct Substitution*

On substituting from Eq. (24) into Eq. (2), we obtain

$$\psi(\mathbf{r}) = \frac{1}{4\pi} \int_{\partial D} \mathrm{d}S'' \left[\frac{\partial}{\partial n''} \rho_{\mathrm{ME}}(\mathbf{r}'') \times \int \mathrm{d}^3 r' \, G_0(\mathbf{r}, \mathbf{r}') G_0(\mathbf{r}', \mathbf{r}'') - \rho_{\mathrm{ME}}(\mathbf{r}'') \frac{\partial}{\partial n''} \int \mathrm{d}^3 r' \, G_0(\mathbf{r}, \mathbf{r}') G_0(\mathbf{r}', \mathbf{r}'') \right]. \quad (30)$$

On comparing Eqs. (28) and (30), we note that, to show that Eq. (28) holds, one needs only to show, first, that the operators $\partial/\partial k$ and $\partial/\partial n'$ acting on $G_0(\mathbf{r}, \mathbf{r}')$ commute and, second, that

$$\frac{2\pi}{k} \frac{\partial}{\partial k} G_0(\mathbf{r}, \mathbf{r}'') = \int \mathrm{d}^3 r' \, G_0(\mathbf{r}, \mathbf{r}') G_0(\mathbf{r}', \mathbf{r}''). \quad (31)$$

By using the Fourier transform representation[16]

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi^2} \int \mathrm{d}^3 K' \frac{\exp[i\mathbf{K}' \cdot (\mathbf{r} - \mathbf{r}')]}{K'^2 - k^2}, \quad (32)$$

one verifies that $(\partial/\partial k)(\partial/\partial n')G_0(\mathbf{r}, \mathbf{r}') = (\partial/\partial n') \times (\partial/\partial k)G_0(\mathbf{r}, \mathbf{r}')$ and also that

$$\frac{2\pi}{k} \frac{\partial}{\partial k} G_0(\mathbf{r}, \mathbf{r}'') = \frac{2}{\pi} \int \mathrm{d}^3 K' \frac{\exp[i\mathbf{K}' \cdot (\mathbf{r} - \mathbf{r}'')]}{(K'^2 - k^2)^2} = \int \mathrm{d}^3 r' \, G_0(\mathbf{r}, \mathbf{r}') G_0(\mathbf{r}', \mathbf{r}''), \quad (33)$$

which immediately confirms Eq. (28).

4. EXAMPLES AND SOURCE/FIELD DECOMPOSITIONS

A. Source in One-Dimensional Space

1. *General Case*

To develop further the discussion above, we consider next the 1D counterpart of the formulation in Subsection 2.A. The field $\Psi(x)$ radiated by a source $\rho(x)$ confined within the interval $[-a, a]$, where

$$(\mathrm{d}^2/\mathrm{d}x^2 + k^2)\Psi(x) = -\rho(x), \quad (34)$$

is given by[8]

$$\Psi(x) = \frac{i}{2k} \int_{-a}^{a} \mathrm{d}x' \rho(x') \exp(ik|x - x'|), \quad (35)$$

so that

$$\Psi(x) = \begin{cases} F^+ \exp(ikx) & \text{if } x > a \\ F^- \exp(-ikx) & \text{if } x < -a \end{cases}, \quad (36)$$

where

$$F^+ = \frac{i}{2k}\int_{-a}^{a} \mathrm{d}x'\,\rho(x')\exp(-ikx'),$$

$$F^- = \frac{i}{2k}\int_{-a}^{a} \mathrm{d}x'\,\rho(x')\exp(ikx'). \qquad (37)$$

The inverse source problem in 1D space can thus be stated as being that of deducing $\rho(x)$ from knowledge of the forward and backward plane-wave amplitudes F^+ and F^-, respectively. The ME solution can be evaluated by means of the 1D analog of the formulation in Subsection 2.A. We thus define the Hilbert space U of L^2 functions of $x \in \Re$ localized within the interval $[-a,\ a]$ and assign to it the inner product

$$(\rho, \rho')_U = \int_{-a}^{a} \mathrm{d}x'\,\rho^*(x')\rho'(x'). \qquad (38)$$

We also define the data space V of square-summable vectors $\mathbf{F} = [F^+ \quad F^-]$, where $|F^+|^2 + |F^-|^2 < \infty$, and assign to it the inner product

$$(\mathbf{F}, \mathbf{F}')_V = (F^+)^* F'^+ + (F^-)^* F'^-. \qquad (39)$$

The linear mapping $T: U \to V$ of the space U into the space V is defined from Eqs. (37) by

$$(T\rho)^{\pm} = \frac{i}{2k}\int \mathrm{d}x'\,M(x')\rho(x')\exp(\mp ikx'), \qquad (40)$$

where $M(x) = 1$ if $|x| \leq a$ and is zero elsewhere. The adjoint $T^\dagger$ of T, defined by $(T\rho, \mathbf{F})_V = (\rho, T^\dagger\mathbf{F})_U$, is found from Eqs. (38)–(40) to be given by

$$(T^\dagger\mathbf{F})(x) = -\frac{i}{2k}M(x)[F^+\exp(ikx) + F^-\exp(-ikx)], \qquad (41)$$

which is identified to be a free-field plane-wave expansion truncated within the source's support. By analogy with Eqs. (15) and (16), the ME solution $\rho_{\mathrm{ME}}(x)$ to the 1D inverse source problem is defined by

$$\rho_{\mathrm{ME}} = T^\dagger\tilde{\mathbf{F}}, \qquad (42)$$

where

$$\tilde{\mathbf{F}} = (TT^\dagger)^{-1}\mathbf{F}. \qquad (43)$$

After some manipulations, we obtain the following from Eqs. (40)–(43):

$$\rho_{\mathrm{ME}}(x) = \frac{ikM(x)}{a[\mathrm{sinc}^2(2ka) - 1]} \times \{[F^+ - F^-\,\mathrm{sinc}(2ka)]\exp(ikx) + [F^- - F^+\,\mathrm{sinc}(2ka)]\exp(-ikx)\}, \qquad (44)$$

where $\mathrm{sinc}(\cdot) \equiv \sin(\cdot)/(\cdot)$. We find from Eqs. (37) and (44) and l'Hôpital's rule that

$$\rho_{\mathrm{ME}}(x) \sim \bar{\rho}M(x) \qquad \text{as} \quad k \to 0, \qquad (45)$$

where $\bar{\rho}$ is the mean of $\rho(x)$ as computed over the interval $[-a,\ a]$, i.e.,

$$\bar{\rho} = \frac{1}{2a}\int_{-a}^{a} \mathrm{d}x'\,\rho(x').$$

The field $\Psi_{\mathrm{ME}}(x)$ produced by the ME source $\rho_{\mathrm{ME}}(x)$ can be evaluated with the aid of the 1D version of Theorem 2. In particular, $\Psi_{\mathrm{ME}}(x)$ can be shown to obey the iterated homogeneous Helmholtz equation

$$(\mathrm{d}^2/\mathrm{d}x^2 + k^2)^2\Psi_{\mathrm{ME}}(x) = 0 \qquad \text{if } |x| < a,$$

so that

$$\Psi_{\mathrm{ME}}(x) = A\cos(kx) + B\sin(kx) + Cx\sin(kx) + Dx\cos(kx) \qquad \text{if } |x| < a \qquad (46)$$

or

$$\Psi_{\mathrm{ME}}(x) = A'\exp(ikx) + B'\exp(-ikx) + C'x\exp(ikx) + D'x\exp(-ikx) \qquad \text{if } |x| < a, \qquad (47)$$

where $A,\ B,\ C$, and D or $A',\ B',\ C'$, and D' are coefficients that need to be evaluated. To solve for the coefficients $A,\ B,\ C$, and D or $A',\ B',\ C'$, and D', we note that the ME source $\rho_{\mathrm{ME}}(x)$ in Eq. (44) is bounded and piecewise continuous; this automatically forces $\Psi_{\mathrm{ME}}(x)$ to be everywhere continuous (see Appendix A of Ref. 19). It follows from Eqs. (34) and (36) that one can evaluate the coefficients $A,\ B,\ C$, and D or $A',\ B',\ C'$, and D' from the boundary conditions $\Psi_{\mathrm{ME}}(a) = F^+\exp(ika)$ and $\Psi_{\mathrm{ME}}(-a) = F^-\exp(ika)$ and the fact that $(\mathrm{d}^2/\mathrm{d}x^2 + k^2)\Psi_{\mathrm{ME}}(x) = -\rho_{\mathrm{ME}}(x)$. After some manipulations the coefficients C' and D' are found to be

$$C' = \frac{F^-\,\mathrm{sinc}(2ka) - F^+}{2a[\mathrm{sinc}^2(2ka) - 1]},$$

$$D' = \frac{F^- - F^+\,\mathrm{sinc}(2ka)}{2a[\mathrm{sinc}^2(2ka) - 1]}. \qquad (48)$$

On the other hand, the coefficients A' and B' are found to be given in terms of C' and D' by

$$A' = \frac{1}{2i\sin(2ka)}[F^+\exp(2ika) - F^- - 2aC'\cos(2ka) - 2aD'],$$

$$B' = \frac{1}{2i\sin(2ka)}[F^-\exp(2ika) - F^+ + 2aC' + 2aD'\cos(2ka)]. \qquad (49)$$

Both the formulation in 1D space presented above and expression (44) for the corresponding ME solution are new. Expressions (47)–(49) specifying the field $\Psi_{\mathrm{ME}}(x)$ produced by the ME source $\rho_{\mathrm{ME}}(x)$ are also new. Equations (44) and (47)–(49) can be used along with Eqs. (35) and (37) and the discussion in Subsection 2.A.2 to uniquely decompose any L^2 source $\rho(x)$ localized within the interval $[-a,a]$ and its field $\Psi(x)$ into the sum of a radiating and a NR part. To illustrate further these results and some of their consequences, we consider next the special case of a homogeneous source.

2. *Special Case: Homogeneous Source*

If $\rho(x)$ is an even function, then $F^+ = F^-$ and Eq. (44) yields

$$\rho_{\mathrm{ME}}(x) = -\frac{2ik}{a[\operatorname{sinc}(2ka) + 1]} F^{+} M(x) \cos(kx). \quad (50)$$

For a homogeneous source $\rho(x) = M(x)$, we obtain

$$F^{+} = \frac{i}{k^2} \sin(ka)$$

from Eq. (37), so that from Eq. (50) we have

$$\rho_{\mathrm{ME}}(x) = \eta M(x) \cos(kx), \quad (51)$$

where

$$\eta = \frac{2 \operatorname{sinc}(ka)}{\operatorname{sinc}(2ka) + 1}.$$

The NR part of the homogeneous source $\rho(x) = M(x)$ is then

$$\rho_{\mathrm{NR}}(x) = M(x)[1 - \eta \cos(kx)]. \quad (52)$$

Equations (51) and (52) establish the unique decomposition of the homogeneous source $\rho(x) = M(x)$ into its radiating and NR parts, $\rho_{\mathrm{ME}}(x)$ and $\rho_{\mathrm{NR}}(x)$, respectively. We consider next the corresponding field decomposition.

For $\rho(x) = M(x)$, expression (35) yields

$$\Psi(x) = \frac{i}{2k}\left\{ \int_{-a}^{x} \mathrm{d}x' \exp[ik(x - x')] + \int_{x}^{a} \mathrm{d}x' \exp[-ik(x - x')] \right\} \quad \text{if } |x| \le a$$
$$= \frac{1}{k^2}[\exp(ika)\cos(kx) - 1] \quad \text{if } |x| \le a. \quad (53)$$

On the other hand, the field $\Psi_{\mathrm{ME}}(x)$ produced by the radiating part of the homogeneous source $\rho(x) = M(x)$, $\rho_{\mathrm{ME}}(x)$, is evaluated by using Eqs. (46) and (51) and the boundary conditions $\Psi_{\mathrm{ME}}(a) = \Psi_{\mathrm{ME}}(-a) = (i/k^2)\sin(ka)\exp(ika)$. We obtain, for $|x| \le a$,

$$\Psi_{\mathrm{ME}}(x) = \frac{\eta}{2k}\{ia[\exp(ika)\operatorname{sinc}(ka) + 1]\cos(kx) - x \sin(kx)\}. \quad (54)$$

Alternatively, by using Eq. (35) with $\rho(x) = \rho_{\mathrm{ME}}(x)$ and Eq. (51), one obtains, for $|x| \le a$,

$$\Psi_{\mathrm{ME}}(x) = \frac{i\eta}{2k}\left\{ \int_{-a}^{x} \mathrm{d}x' \cos(kx')\exp[ik(x - x')] + \int_{x}^{a} \mathrm{d}x' \cos(kx')\exp[-ik(x - x')] \right\}$$
$$= \frac{\eta}{4k^2}[2ika + \exp(2ika) - 1]\cos(kx) - \frac{\eta}{2k} x \sin(kx). \quad (55)$$

Expression (55) can be shown, after some algebra, to reduce to our previous result [Eq. (54)]. This confirms the validity of Eq. (54) and of our new procedure leading to that result.

The fields $\Psi(x)$ and $\Psi_{\mathrm{ME}}(x)$ are identical for $|x| > a$ and are given by Eq. (36) with $F^{+} = F^{-} = (i/k^2)\sin(ka)$. The NR field $\Psi_{\mathrm{NR}}(x) = \Psi(x) - \Psi_{\mathrm{ME}}(x)$ produced by the NR part $\rho_{\mathrm{NR}}(x)$ of the homogeneous source $\rho(x) = M(x)$ vanishes for $|x| > a$. On the other hand, for $|x| \le a$, the NR field $\Psi_{\mathrm{NR}}(x) = \Psi(x) - \Psi_{\mathrm{ME}}(x)$ is explicitly defined by Eqs. (53) and (54). We have thus carried out, explicitly, the unique decomposition of the homogeneous source $\rho(x) = M(x)$ and its field $\Psi(x)$ into their radiating and NR parts, $\rho_{\mathrm{ME}}(x)$ and $\rho_{\mathrm{NR}}(x)$, and $\Psi_{\mathrm{ME}}(x)$ and $\Psi_{\mathrm{NR}}(x)$, respectively. The following remarks are in order:

1. The ME source in Eqs. (50) and (51) is recognized as being a standing wave truncated within the source's support $[-a, a]$; it possesses symmetry with respect to the origin to accommodate for that of $\rho(x)$, from which it was derived.
2. The field $\Psi_{\mathrm{ME}}(x)$ produced by the radiating part of this source, $\rho_{\mathrm{ME}}(x)$, obeys an iterated homogeneous Helmholtz equation, the solution of which has been given explicitly. The field $\Psi_{\mathrm{ME}}(x)$ associated with $\rho_{\mathrm{ME}}(x)$ was uniquely determined by the field data (i.e., F^{+} and F^{-}), as expected.
3. It follows from Eq. (51) that $\rho_{\mathrm{ME}}(x) = 0$ if $ka = n\pi$, $n = 1, 2, \ldots$, i.e., a homogeneous source $\rho(x) = M(x)$ oscillating at those frequencies is purely NR. Alternatively, by computing from Eq. (51) the inner product

$$(\rho_{\mathrm{ME}}, \rho)_U = 2a\eta \operatorname{sinc}(ka) = \frac{4a \operatorname{sinc}^2(ka)}{\operatorname{sinc}(2ka) + 1}, \quad (56)$$

one confirms that for $ka = n\pi$, $n = 1, 2, \ldots$, the source $\rho(x) = M(x)$ has no radiating part. Yet another way of verifying the existence of these NR frequencies consists in evaluating from Eq. (53) the boundary values $\Psi(a)$ and $\Psi(-a)$ under the NR condition $ka = n\pi$, $n = 1, 2, \ldots$. We obtain from Eq. (53) the following expression for the NR field corresponding to the n^{th} NR mode:

$$\Psi_{\mathrm{NR}}(x) = \left(\frac{a}{n\pi}\right)^2 [(-1)^n \cos(n\pi x/a) - 1] \quad \text{if } |x| \le a. \quad (57)$$

Thus for these NR modes $\Psi_{\mathrm{NR}}(a) = \Psi_{\mathrm{NR}}(-a) = 0$, which automatically guarantees, in view of Eq. (36), the vanishing of the corresponding NR field $\Psi_{\mathrm{NR}}(x)$ for $|x| > a$ [since then $F^{+} = \Psi_{\mathrm{NR}}(a)\exp(-ika) = 0$ and $F^{-} = \Psi_{\mathrm{NR}}(-a)\exp(-ika) = 0$]. The NR condition for homogeneous sources given above and expression (57) for the NR field associated with a NR homogeneous source are identical with those found by Berry and co-workers [see Eqs. (12) and (13) of Ref. 6]. The projection $(\rho_{\mathrm{ME}}, \rho)_U$ in Eq. (56) exhibits a damped oscillatory dependence on ka: It exhibits local maxima at $ka \simeq (n + 1/2)\pi$, $n = 1, 2, \ldots$ [the source's radiating behavior is (locally) enhanced at those frequencies]. It decays rapidly for $ka \gtrsim \pi$. In general, as ka increases, the source $\rho(x) = M(x)$ becomes predominantly NR. In particular, its radiating part $\rho_{\mathrm{ME}}(x)$ can be shown from Eq. (51) to decay asymptotically to zero as $k \to \infty$. That there are no NR homogeneous sources for $ka < \pi$ is to be expected, since then the size of the homogeneous source, relative to the wavelength $\lambda = 2\pi/k$, prevents the required cancellation—through destructive interference—of the wave fronts produced by the different source elements. The extension of the smallest NR homogeneous source is $2a = \lambda$, for which the source's size coincides with the wavelength λ of the field. At high frequencies most of the homogeneous source is NR; only the source elements in the vicinity of the boundaries $x = a$ and $x = -a$ contribute to the field for $|x| > a$. This can be visualized by noting that the homogeneous source $\rho(x) = M(x)$ can be expressed as

$$\rho(x) = M'(x) + M''(x),$$

where $M'(x) = 1$ if $|x| \leq P\lambda/2$ and is zero otherwise and $M''(x) = 1$ if $P\lambda/2 \leq |x| \leq a$ and is zero otherwise, where P is the integer part of $2a/\lambda$, i.e.,

$$P = \begin{cases} 0 & \text{if } 2a < \lambda \\ 1 & \text{if } \lambda \leq 2a < 2\lambda \\ 2 & \text{if } 2\lambda \leq 2a < 3\lambda \end{cases},$$

and so on. The term $M'(x)$ is identified to be a NR source according to the NR condition for homogeneous sources given above. It is a NR homogeneous source of support $[-P\lambda/2, P\lambda/2]$. Thus, for $|x| > a$, the field $\Psi(x)$ produced by $\rho(x) = M(x)$ must be entirely due to $M''(x)$. The ratio $P\lambda/2a$ defines the size of the NR region associated with $M'(x)$ relative to that of the total source $\rho(x) = M(x)$. That ratio goes asymptotically to unity as $\lambda \to 0$, as expected from the discussion above. On the other hand, the length of the strips $[-a, -P\lambda/2]$ and $[P\lambda/2, a]$ contributing to radiation for $|x| > a$ decays as λ decreases.

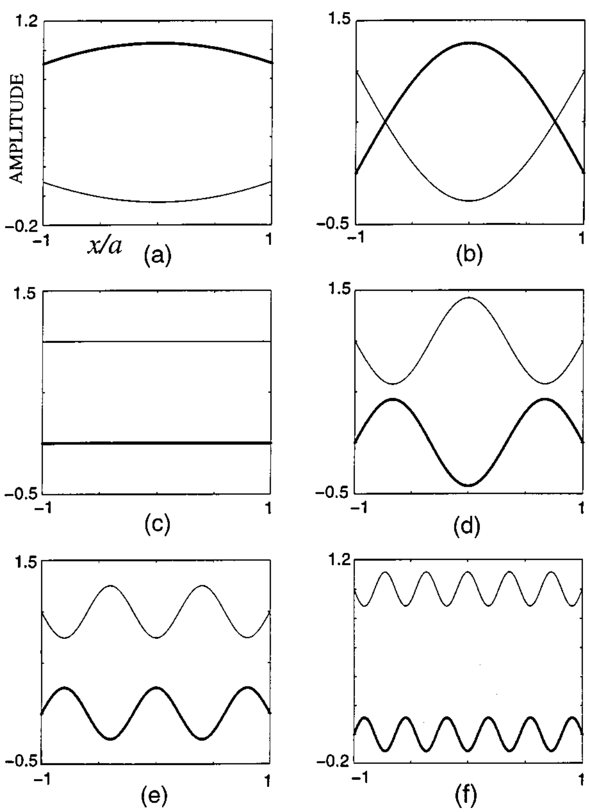

Fig. 1. Radiating part (thick curves) and NR part (thin curves) of the homogeneous source $\rho(x) = M(x)$ versus x/a for (a) $ka = \pi/6$, (b) $ka = \pi/2$, (c) $ka = \pi$ (NR case), (d) $ka = 1.5\pi$, (e) $ka = 2.5\pi$, (f) $ka = 5.5\pi$.

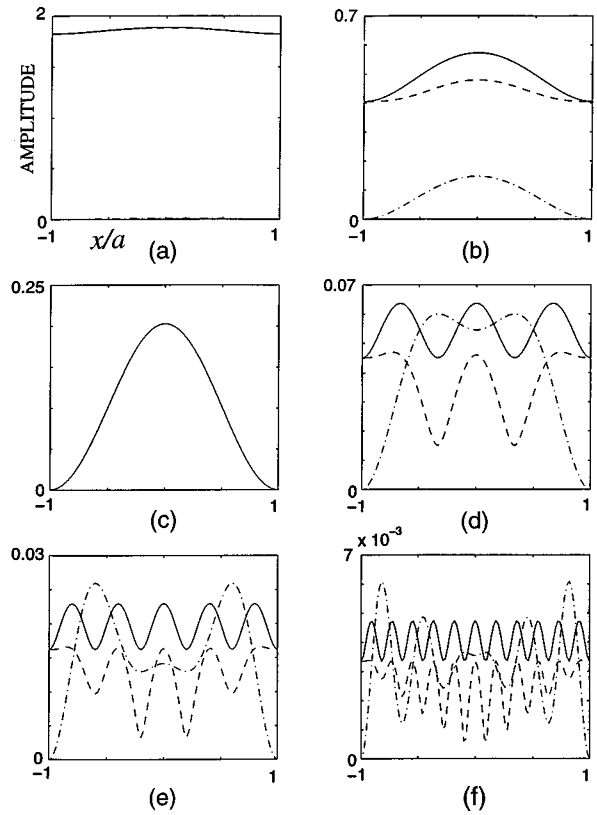

Fig. 2. Field magnitude for $|x| \leq a$ produced by the homogeneous source $\rho(x) = M(x)$ versus x/a (solid curves). Also shown are the magnitudes of the radiating part (dashed curves) and the NR part (dotted–dashed curves) of the total field for $|x| \leq a$. (a) $ka = \pi/6$, (b) $ka = \pi/2$, (c) $ka = \pi$ (NR case), (d) $ka = 1.5\pi$, (e) $ka = 2.5\pi$, (f) $ka = 5.5\pi$.

4. We find from Eq. (51) and l'Hôpital's rule that $\rho_{\mathrm{ME}}(x) \sim M(x)$ as $k \to 0$ [this result follows also from the discussion in relation (45)]. Thus, for $k \simeq 0$, the homogeneous source $\rho(x) = M(x)$ is almost purely radiating.

5. The ME sources $\rho_{\mathrm{ME}}(x)$ in Eqs. (50) and (51) possess compact support in $[-a, a]$ only if $ka = (n + 1/2)\pi$, $n = 0, 1, \ldots$.

Figure 1 shows plots of the spatial profile of the radiating and NR parts of $\rho(x) = M(x)$ for $|x| \leq a$, parameterized by ka. Figure 1(a) corresponds to $ka = \pi/6$ and illustrates the low-frequency nature of the source decomposition wherein the homogeneous source is mostly radiating (see observation 4 above). Figure 1(b) corresponds to $ka = \pi/2$ and illustrates observation 5 above. In the latter case, the radiating part of the homogeneous source vanishes on the boundaries $x = a$ and $x = -a$ and therefore has compact support in $[-a, a]$. Figure 1(c) corresponds to the smallest NR homogeneous source, wherein $ka = \pi$. Figures 1(d), 1(e), and 1(f) correspond to $ka = 1.5\pi$, 2.5π, and 5.5π, respectively. A gradual increase of the NR component of the homogeneous source as ka increases is observed, which is to be expected from observation 3 above. Figure 2 shows plots of the spatial profile of the magnitude of the radiating and NR fields associated with the radiating and NR source components above. Also shown are the total fields (magnitude only). The latter are seen to coincide with the radiating fields $\psi_{\rm ME}(x)$ on the boundaries $x = a$ and $x = -a$, as expected.

B. Spherically Symmetric Source

We consider next the case of a spherically symmetric source $\rho(r)$ contained within the spherical volume τ. This case can be regarded as the 3D counterpart of the 1D problem addressed in Subsection 4.A. In this case Eq. (1) reduces to

$$(\mathrm{d}^2/\mathrm{d}r^2 + k^2)[r\psi(r)] = -4\pi r\rho(r). \tag{58}$$

By using[8]

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{\exp(ik|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} = 4\pi ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l} j_l(kr_<)h_l^{(1)}(kr_>) \times Y_{l,m}(\hat{\mathbf{r}})Y_{l,m}^*(\hat{\mathbf{r}}'),$$

where $r_<$ and $r_>$ denote, respectively, the smaller and the larger of r and r', one finds from Eq. (2) the field $\psi(r)$ produced by the spherically symmetric source $\rho(r)$ to be given by

$$\psi(r) = 4\pi ik\int_0^a \mathrm{d}r'\, r'^2\rho(r')j_0(kr_<)h_0^{(1)}(kr_>), \tag{59}$$

where we have used $Y_{0,0} = 1/\sqrt{4\pi}$ and the orthogonality property of the spherical harmonics (see Ref. 15, p. 681–682). For $r \leqslant a$, Eq. (59) yields

$$\psi(r) = Q(r)\exp(ikr)/r + R(r)\mathrm{sinc}(kr), \tag{60}$$

where

$$Q(r) = 4\pi k^{-1}\int_0^r \mathrm{d}r'\, r'\rho(r')\sin(kr'),$$

$$R(r) = 4\pi\int_r^a \mathrm{d}r'\, r'\rho(r')\exp(ikr'). \tag{61}$$

In deriving Eqs. (60) and (61), we have used $j_0(kr) = \mathrm{sinc}(kr)$ and $h_0^{(1)}(kr) = -i\exp(ikr)/(kr)$ (see Ref. 15, p. 625). On the other hand, for $r > a$, we obtain, from Eq. (59)

$$\psi(r) = Q(a)\exp(ikr)/r, \tag{62}$$

which is identified to be the spherically symmetric component of the multipole expansion (6), i.e.,

$$\psi(r) = \frac{ik}{\sqrt{4\pi}}g_{0,0}h_0^{(1)}(kr) = \frac{g_{0,0}}{\sqrt{4\pi}}\exp(ikr)/r \qquad (r > a), \tag{63}$$

with the relevant multipole moment $g_{0,0}$ given from Eqs. (7) and (8) by

$$g_{0,0} = (4\pi)^{3/2}\int_0^a \mathrm{d}r'\, r'^2\rho(r')j_0(kr') = (4\pi)^{3/2}k^{-1}\int_0^a \mathrm{d}r'\, r'\rho(r')\sin(kr'). \tag{64}$$

1. *Minimum-Energy Solution and Source/Field Decompositions*

The ME solution to the inverse source problem associated with a spherically symmetric source/field is given from Eqs. (8), (17), and (18) by

$$\rho_{\rm ME}(r) = \sqrt{4\pi}g_{0,0}U(a - r)j_0(kr)/\sigma_0^2, \tag{65}$$

where $U(\cdot)$ is Heaviside's step function and

$$\sigma_0^2 = 8\pi^2a^3[j_0^2(ka) - j_{-1}(ka)j_1(ka)] = \frac{8\pi^2a}{k^2}[1 - \mathrm{sinc}(2ka)]. \tag{66}$$

In deriving Eq. (66), we have used $j_1(ka) = [\mathrm{sinc}(ka) - \cos(ka)]/(ka)$ and the recurrence relations of the spherical Bessel functions (see Ref. 15, pp. 626–627).

The field $\psi_{\rm ME}(r)$ produced by $\rho_{\rm ME}(r)$ can be shown from Theorem 2 to obey

$$(\mathrm{d}^2/\mathrm{d}r^2 + k^2)^2[r\psi_{\rm ME}(r)] = 0 \qquad \text{if } r \leqslant a,$$

so that

$$\psi_{\rm ME}(r) = A\,\mathrm{sinc}(kr) + B\frac{\cos(kr)}{r} + C\sin(kr) + D\cos(kr) \qquad \text{if } r \leqslant a, \tag{67}$$

where A, B, C, and D are coefficients that remain to be evaluated. In view of the singularity of the second term in Eq. (67) for $r = 0$, we require that $B = 0$. We require from Eq. (63) that

$$\psi_{\rm ME}(a) = \frac{g_{0,0}}{\sqrt{4\pi}}\exp(ika)/a.$$

We also require from Eq. (58) that $(\mathrm{d}^2/\mathrm{d}r^2 + k^2) \times [r\psi_{\rm ME}(r)] = -4\pi r\rho_{\rm ME}(r)$. These requirements uniquely define the coefficients A, B, C, and D in Eq. (67). After some manipulations one obtains, for $r \leqslant a$,

$$\psi_{\rm ME}(r) = g_{0,0}\left[\frac{1}{\sqrt{4\pi}a}\exp(ika) - 4\pi^{3/2}k^{-2}\sigma_0^{-2}\cos(ka)\right] \times \frac{\mathrm{sinc}(kr)}{\mathrm{sinc}(ka)} + \frac{4\pi^{3/2}g_{0,0}}{k^2\sigma_0^2}\cos(kr). \tag{68}$$

Equations (60), (61), (63)–(66), and (68) apply to any spherically symmetric source $\rho(r)$ and can be used to uniquely decompose the source and its field into their radiating and NR parts. In particular, for a given $\rho(r)$, one can compute from Eq. (64) the relevant multipole moment $g_{0,0}$ and subsequently evaluate the radiating part of the source, $\rho_{\rm ME}(r)$, by using Eqs. (65) and (66). On the other hand, the NR part is $\rho_{\rm NR}(r) = \rho(r) - \rho_{\rm ME}(r)$, which completes the source decomposition. The field $\psi(r)$, on the other hand, can be uniquely decomposed into its radiating and NR parts using Eqs. (60), (61), (63), (64), and (68). For $r > a$, $\psi(r) = \psi_{\mathrm{ME}}(r)$ is defined by Eqs. (63) and (64), and $\psi_{\mathrm{NR}}(r) = 0$. For $r \leqslant a$, $\psi_{\mathrm{ME}}(r)$ is given by Eqs. (64), (66), and (68), whereas the NR field $\psi_{\mathrm{NR}}(r) = \psi(r) - \psi_{\mathrm{ME}}(r)$ can be evaluated by using the expressions for $\psi(r)$ and $\psi_{\mathrm{ME}}(r)$ in Eqs. (60) and (61) and Eqs. (64), (66), and (68), respectively.

2. Special Case: Homogeneous Spherical Source

Finally, we address the 3D counterpart of the results on homogeneous 1D sources presented in Subsection 4.A.2. In particular, we briefly discuss the unique decomposition of the homogeneous spherically symmetric source $\rho(r) = U(a - r)$ and its field $\psi(r)$ into their radiating and NR parts. It follows from Eq. (64) that, for this source,

$$g_{0,0} = (4\pi)^{3/2} k^{-1} \int_0^a \mathrm{d}r'\, r' j_0(kr') = \frac{(4\pi)^{3/2}}{k^2}\left[-a\cos(ka) + \frac{\sin(ka)}{k}\right]. \qquad (69)$$

The radiating part of $\rho(r) = U(a - r)$, $\rho_{\mathrm{ME}}(r)$, is given from Eqs. (65), (66), and (69) by

$$\rho_{\mathrm{ME}}(r) = 2\,\frac{\mathrm{sinc}(ka) - \cos(ka)}{1 - \mathrm{sinc}(2ka)}\, U(a - r)\,\mathrm{sinc}(kr), \qquad (70)$$

while $\rho_{\mathrm{NR}}(r) = U(a - r) - \rho_{\mathrm{ME}}(r)$. We find from expression (70) and l'Hôpital's rule that $\rho_{\mathrm{ME}}(x) \sim U(a$

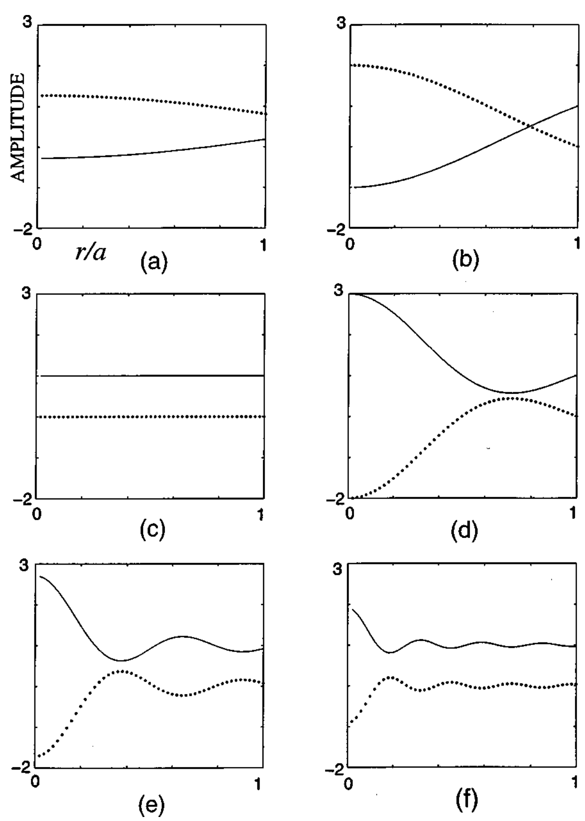

Fig. 3. Radiating part (dotted curves) and NR part (solid curves) of the homogeneous spherical source $\rho(r) = U(a - r)$ versus r/a for (a) $ka = \pi/2$, (b) $ka = \pi$, (c) $ka = 4.493$ (NR case), (d) $ka = 2\pi$, (e) $ka = 12$, (f) $ka = 24$.

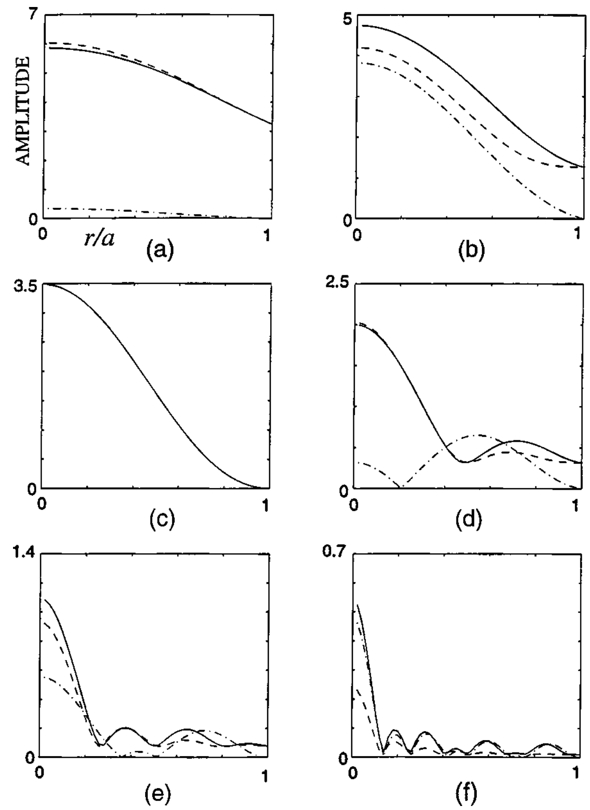

Fig. 4. Field magnitude for $r \leqslant a$ produced by the homogeneous spherical source $\rho(r) = U(a - r)$ versus r/a (solid curves). Also shown are the magnitudes of the radiating part (dashed curves) and the NR part (dotted–dashed curves) of the total field for $r \leqslant a$. (a) $ka = \pi/2$, (b) $ka = \pi$, (c) $ka = 4.493$ (NR case), (d) $ka = 2\pi$, (e) $ka = 12$, (f) $ka = 24$.

$- r)$ as $k \to 0$, as expected (for $k \simeq 0$ the homogeneous source is almost purely radiating).

For $r \leqslant a$ the field $\psi(r)$ generated by the homogeneous source $\rho(r) = U(a - r)$ is found from Eqs. (60) and (61) to be given by

$$\psi(r) = \frac{4\pi}{k^2}[(1 - ika)\exp(ika)\mathrm{sinc}(kr) - 1]. \qquad (71)$$

For $r \leqslant a$ the field $\psi_{\mathrm{ME}}(r)$ produced by the radiating part of $\rho(r)$, $\rho_{\mathrm{ME}}(r)$, is defined by Eq. (68), with $g_{0,0}$ given by Eq. (69). The remaining details of the source/field decompositions for this special case are contained in the paragraph that follows Eq. (68) and will not be repeated here. Figures 3 and 4 show plots of the radiating and NR parts of the homogeneous source $\rho(r) = U(a - r)$ and its field $\psi(r)$ (magnitude only) versus r/a, parameterized by ka. Figure 4 also contains plots of the magnitude of the total field $\psi(r)$. Some observations analogous to those given at the end of Subsection 4.A are in order. The NR condition $g_{0,0} = 0$ is seen from Eq. (69) to reduce to

$$\mathrm{sinc}(ka) = \cos(ka), \qquad (72)$$

i.e., $\rho(r) = U(a - r)$ is purely NR if $ka \simeq 4.493, 7.725, 10.904, \ldots$ (see also Ref. 7, p. 5). The field that is due to a NR homogeneous source is found from Eqs. (71) and (72) to be given by

$$\psi_{\mathrm{NR}}(r) = \frac{4\pi}{k^2}[\mathrm{sinc}(kr)/\cos(ka) - 1], \tag{73}$$

which is the result [Eq. (5.8)] of Kim and Wolf.[7] The spatial profiles of the smallest NR homogeneous source and its field are shown in Figs. 3(c) and 4(c), respectively. The ME source $\rho_{\mathrm{ME}}(r)$ in Eq. (65) possesses compact support in the spherical volume τ only if $j_0(ka) = 0$, i.e., for $ka = n\pi$, $n = 1,2,\ldots$ [see Figs. 3(b) and 3(d)]. Under the latter condition, the ME source $\rho_{\mathrm{ME}}(r)$ in Eq. (65) is also seen to possess a nonzero normal derivative on the boundary $r = a$ (since no zero of $j_0(\cdot)$ is also a zero of $j_0'(\cdot) = -j_1(\cdot)$ [see Eqs. (11.148) and (11.154) of Ref. 15]). This is to be expected from Theorem 3. Other observations follow readily from the formulation above and Figs. 3 and 4 by analogy with those given for their 1D counterparts in Subsection 4.A.

In summary, we have presented the 3D analogs of the 1D case results given in Subsection 4.A. The spherically symmetric case results also illustrate Theorems 1–3 of Subsection 2.B and, in general, the source-inversion approach for 3D sources presented in Section 2. The procedure employed to derive some of our results for the spherically symmetric case resembles that corresponding to the 1D case if one makes the substitutions $\rho(x) \rightarrow 4\pi r\rho(r)$ and $\Psi(x) \rightarrow r\psi(r)$ and uses the appropriate boundary conditions for the spherical problem.

5. CONCLUDING REMARKS

In contrast to the approach of previous contributions,[2–4] ME sources were characterized here from the points of view of both direct and inverse source problems. In Subsection 2.A and Section 4, we applied standard linear inversion theory to the inverse source problem in 3D and 1D spaces and computed the ME solution to this problem. Among other results, ME sources of specified support were found to be waves truncated within that support (Theorem 1). It then follows that the primary sources of wave radiation are also waves (refer to Theorem 1 and Subsection 2.A.2). It also follows that the fields produced by ME sources obey an iterated homogeneous Helmholtz equation (Theorem 2). It was also shown that any L^2 source of compact support D that possesses a continuous normal derivative on the boundary ∂D of D must possess a NR part (Theorem 3). This previously unknown result speaks about an intrinsically unobservable component associated with a broad class of physically relevant sources. In Section 3 we developed new Green-function representations of ME sources and their fields. In Section 4 we addressed the 1D inverse source problem and characterized in detail the radiation properties of homogeneous 1D sources and spherically symmetric sources and of their fields. Some of our results on homogeneous sources can be extended to nonhomogeneous ones. In particular, the source's length or the source's diameter in Subsections 4.A and 4.B, respectively, can represent, in practice, a characteristic dimension of the source or scatterer (not necessarily its size), e.g., a lattice constant in a periodic structure, a correlation length in a random source, etc.

APPENDIX A: SOLUTION OF EQUATION (4)

Our first task will be to determine the solution of the partial differential equation

$$(\nabla^2 + k_1{}^2)(\nabla^2 + k_2{}^2)\psi(\mathbf{r}) = 0 \tag{A1}$$

in a volume region D in terms of the values of $\psi(\mathbf{r})$ and $\nabla^2\psi(\mathbf{r})$ on the closed surface ∂D that bounds the volume D. The solution of Eq. (A1) was obtained in Ref. 18 by using the (interior problem) Green function that obeys homogeneous Dirichlet boundary conditions on the boundary ∂D of the volume D. Here we use the Green function that obeys Sommerfeld's radiation condition instead. Following the lines of Ref. 18, we define

$$\psi_1(\mathbf{r}) = (\nabla^2 + k_2{}^2)\psi(\mathbf{r}), \qquad \psi_2(\mathbf{r}) = (\nabla^2 + k_1{}^2)\psi(\mathbf{r}), \tag{A2}$$

where $k_1{}^2 \neq k_2{}^2$, so that

$$\psi(\mathbf{r}) = \frac{[(\nabla^2 + k_1{}^2) - (\nabla^2 + k_2{}^2)]\psi(\mathbf{r})}{k_1{}^2 - k_2{}^2} = \frac{\psi_1(\mathbf{r}) - \psi_2(\mathbf{r})}{k_2{}^2 - k_1{}^2} \qquad (k_1{}^2 \neq k_2{}^2). \tag{A3}$$

Now, in view of Eqs. (A1) and (A2),

$$(\nabla^2 + k_1{}^2)\psi_1(\mathbf{r}) = 0 \qquad (\nabla^2 + k_2{}^2)\psi_2(\mathbf{r}) = 0. \tag{A4}$$

Let $G_0^{(1)}(\mathbf{r}, \mathbf{r}')$ and $G_0^{(2)}(\mathbf{r}, \mathbf{r}')$ be the Green functions satisfying the equations

$$(\nabla^2 + k_i{}^2)G_0^{(i)}(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}'), \qquad i = 1, 2, \tag{A5}$$

and Sommerfeld's radiation condition. Now the solution to Eqs. (A4) may be expressed in the form

$$\psi_1(\mathbf{r}) = \frac{1}{4\pi}\int_{\partial D} \mathrm{d}S'\left[G_0^{(1)}(\mathbf{r}, \mathbf{r}')\frac{\partial}{\partial n'}(\nabla'^2 + k_2{}^2)\psi(\mathbf{r}') - (\nabla'^2 + k_2{}^2)\psi(\mathbf{r}')\frac{\partial}{\partial n'}G_0^{(1)}(\mathbf{r}, \mathbf{r}')\right],$$

$$\psi_2(\mathbf{r}) = \frac{1}{4\pi}\int_{\partial D} \mathrm{d}S'\left[G_0^{(2)}(\mathbf{r}, \mathbf{r}')\frac{\partial}{\partial n'}(\nabla'^2 + k_1{}^2)\psi(\mathbf{r}') - (\nabla'^2 + k_1{}^2)\psi(\mathbf{r}')\frac{\partial}{\partial n'}G_0^{(2)}(\mathbf{r}, \mathbf{r}')\right]. \tag{A6}$$

The solution of Eq. (A1) subject to the prescribed values of $\psi(\mathbf{r})$ and $\nabla^2\psi(\mathbf{r})$ on ∂D is given by Eq. (A3), with $\psi_1(\mathbf{r})$ and $\psi_2(\mathbf{r})$ defined in Eqs. (A6). The solution of Eq. (4) with specified boundary conditions $\psi(\mathbf{r})$ and $\nabla^2\psi(\mathbf{r})$ on ∂D is given by the limit, as $k_2 \rightarrow k_1 = k$, of the solution of Eq. (A1) that obeys the same boundary conditions.[18] Thus, on using l'Hôpital's rule, we obtain

$$\psi(\mathbf{r}) = \lim_{k_2\rightarrow k_1 = k}\frac{\psi_2(\mathbf{r}) - \psi_1(\mathbf{r})}{k_1{}^2 - k_2{}^2} = \frac{1}{2k}\lim_{k_2\rightarrow k_1 = k}\left[\frac{\partial}{\partial k_2}\psi_2(\mathbf{r}) - \frac{\partial}{\partial k_2}\psi_1(\mathbf{r})\right]. \tag{A7}$$

Finally, on carrying out the operations in Eq. (A7), one obtains from Eqs. (A6) the result [Eq. (27)].

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