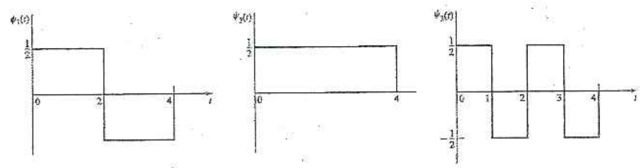
ECE (435/535a) Spring 2016 Midterm 2
April 5

Name: Solution Signature: ____________

Instructions: Answer all questions and show all work. Answers which are not justified with appropriate work will receive 0 points. Students who cheat will receive zero points on the exam and will be subject to the university's disciplinary procedure for academic dishonesty. Cheating includes, but is not limited to, collaborating of conferring in any way with anyone. Use of the internet is strictly forbidden. Your signature above attests that you are in compliance with these rules.

1. Use the orthonormal waveforms shown in the figure below to approximate the function $x(t) = \cos(\frac{\pi t}{4})$, over the interval $0 \le t \le 4$ by the linear combination $\hat{x}(t) = \sum_{n=1}^{3} c_n \psi_n(t)$.



   (a) (15 points) Determine the expansion coefficients $\{c_n\}$ that minimize the mean-square approximation error $\mathcal{E} = \int_0^4 (x(t) - \hat{x}(t))^2 dt$.
   (**Hints:** $\int \sin(x)dx = -\cos(x)$, $\int \cos(x)dx = \sin(x)$ and $\cos^2(x) = \frac{1+\cos(2x)}{2}$.)

$\mathcal{E}$ is minimized when ever is orthogonal to set of orthonormal basis ϕ_n, $n=1,2,3$.

So,

$$c_n = \int_0^4 x(t)\,\phi_n(t)\,dt \quad , \quad n=1,2,3$$

We have:

$$c_1 = \int_0^4 x(t)\,\phi_1(t)\,dt = \int_0^2 \tfrac{1}{2}\cos\left(\tfrac{\pi t}{4}\right)dt + \tfrac{1}{2}\int_2^4 (-\tfrac{1}{2})\cos(\pi t/4)\,dt = 4/\pi$$

$$c_2 = \int_0^4 x(t)\,\phi_2(t)\,dt = \int_0^4 \tfrac{1}{2}\cos\tfrac{\pi t}{4}\,dt = \tfrac{1}{2}\,(4/\pi)\sin \pi t/4 \Big|_0^4 = 0$$

$$c_3 = \int_0^4 x(t)\,\phi_3(t)\,dt = \frac{4\sqrt{2} - 4}{\pi}$$

Problem	Points	Student's Score
1	25	
2	40	
3	35	
Total:	100	

(b) (10 points) Determine the residual mean square error $\mathcal{E}_{\min}$.

$$\mathcal{E}_{min} = \underbrace{\int_{-\infty}^{+\infty} |x(t)|^2 dt}_{\text{I}} - \underbrace{\sum_{i=1}^{3} |c_i|^2}_{\text{II}}$$

$$\text{I}: \int_{-\infty}^{+\infty} \cos^2\frac{\pi t}{4}\, dt = \int_0^4 \left(\tfrac{1}{2} + \tfrac{1}{2}\cos\frac{\pi t}{2}\right) dt = 2$$

$$\text{II} \quad \sum_{i=1}^{3} |c_i|^2 = (4/\pi)^2 + 0^2 + \left(\frac{4\sqrt{2}-4}{\pi}\right)^2$$

$$\mathcal{E}_{min} = 2 - \frac{16}{\pi^2} - \frac{(4\sqrt{2}-4)^2}{\pi^2} \quad \blacksquare$$

2. (PSK Modulation)

(a) (5 points) Write an an expression for a set of M-ary PSK signal waveforms assuming that the $g(t)$ is a baseband pulse shape, ω_c is the career frequency and T is the symbol period.

$$u_m(t) = g(t)\cos\left(2\pi f_c t + \frac{2\pi m}{M}\right), \quad m = 0, 1, \ldots, M-1 \quad \text{and} \quad 0 \le t \le T$$

$$= g(t) A_{mc} \cos(2\pi f_c t) - g(t) A_{ms} \sin(2\pi f_c t)$$

where $A_{mc} = \cos\left(\frac{2\pi m}{M}\right)$, $A_{ms} = \sin\left(\frac{2\pi m}{M}\right)$ for $m = 0, 1, \ldots, M-1$

(b) (5 points) Write an expression for the signal-space representaion of the above waveforms.

PSK can geometrically be represented as two-dimensional vectors with components $\sqrt{E_s}\cos\left(\frac{2\pi m}{M}\right)$ and $\sqrt{E_s}\sin\left(\frac{2\pi m}{M}\right)$

$$\underline{s_m} = \left(\sqrt{E_s}\cos\left(\frac{2\pi m}{M}\right), \sqrt{E_s}\sin\left(\frac{2\pi m}{M}\right)\right) \quad m = 0, 1, \ldots, M-1)$$

(c) (10 points) What are the orthonormal basis functions? How many of them are there?

The orthogonal basis functions for PSK are

$$\psi_1(t) = \sqrt{\frac{2}{E_s}}\, g_T(t) \cos(2\pi f_c t) \quad \text{and}$$

$$\psi_2(t) = \sqrt{\frac{2}{E_s}}\, g_T(t) \sin(2\pi f_c t).$$

(d) (15 points) Find the Euclidean distance between any two signal points in the constellation.

The Euclidean distance (distance between two signal points s_m, s_n) is

$$d_{m,n} = \sqrt{\|s_m - s_n\|^2}$$

$$= \sqrt{2E_s\left(1 - \cos\frac{2\pi}{M}(m-n)\right)}.$$

(e) (5 points) Find the minimum Euclidean distance.

The minimum Euclidean distance (distance between two adjacent points) is

$$d_{min} = \sqrt{2E_s\left(1 - \cos\frac{2\pi}{M}\right)}$$

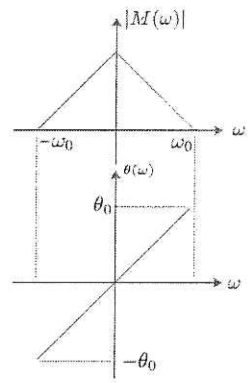
3. (Hilbert Transform)

(a) (10 points) Define the Hilbert transform of a signal $m(t)$.

The hilbert transform of $m(t)$ is defined as

$$\hat{m}(t) = m(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{m(\tau)}{t-\tau} \, d\tau$$

(b) (25 points) The spectrum ($M(\omega) = |M(\omega)|e^{j\theta(\omega)}$) of the signal $m(t)$ is shown below.



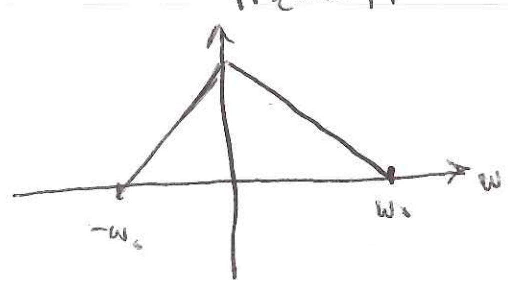
Plot the spectrum of the signal $\hat{m}(t)$ which is equal to the Hilbert transform of $m(t)$.

$$F\{\hat{m}(t)\} = F\{m(t)\} \cdot F\{1/\pi t\} = M(\omega) \cdot (-j\,\mathrm{sgn}(\omega))$$

$$|F\{\hat{m}(t)\}| = |M(\omega)| \cdot |-j\,\mathrm{sgn}(\omega)| = |M(\omega)|$$

$$\measuredangle F\{\hat{m}(t)\} = \measuredangle M(\omega) + \measuredangle(-j\,\mathrm{sgn}(\omega)) = \begin{cases} \measuredangle M(\omega) - \pi/2, & \omega \geq 0 \\ \measuredangle M(\omega) + \pi/2, & \omega < 0 \end{cases}$$

$|F\{\hat{m}(t)\}|$



$\measuredangle F\{\hat{m}(t)\}$