ECE (435/535a) Spring 2016 Midterm
March 8

Name: Solution Signature: ____________________

Instructions: Answer all questions and show all work. Answers which are not justified with appropriate work will receive 0 points. Students who cheat will receive zero points on the exam and will be subject to the university's disciplinary procedure for academic dishonesty. Cheating includes, but is not limited to, collaborating of conferring in any way with anyone. Use of the internet is strictly forbidden. Your signature above attests that you are in compliance with these rules.

1. Source Coding
   (a) (20 points) Design a *ternary* Huffman code for a source with symbol probabilities $p = \{\frac{1}{21}, \frac{2}{21}, \frac{3}{21}, \frac{4}{21}, \frac{5}{21}, \frac{6}{21}\}$.
   Hint: $\log_a x = \dfrac{\log_b x}{\log_b a}$.
   (b) (15 points) Is the average codeword length for such a source larger than the source entropy?

(a)

Codeword	symbol			
1	x_1 $6/21$	$6/21$	$10/21$	
2	x_2 $5/21$	$5/21$	$6/21$	
00	x_3 $4/21$	$4/21$	$5/21$	
01	x_4 $3/21$	$3/21$		
020	x_5 $2/21$	$3/21$		
021	x_6 $1/21$			
022	x_7 $0/21$			

Huffman code $= \{1, 2, 00, 01, 020, 021\}$

(b) $\bar{l} = \sum_i p_i l_i = \cdots = \frac{34}{21} = 1.62$ ternary symbols.

According to the Thm in lecture notes, $H(X) \le \bar{l}$

i.e, $H(X) \le 1.62$

Problem	Points	Student's Score
1	35	
2	35	
3	30	
4	20	
5	10	
Total:	130	

2. Consider an additive white Gaussian channel with output Y, where Y is the sum of the input X and the noise Z, $Y = X + Z$. The input X and the noise Z are drawn from Gaussian distributions $\mathcal{N}(0, \sigma_X^2)$ and $\mathcal{N}(0, \sigma_N^2)$ respectively.
   (a) (5 points) Find the distribution of the output Y.
   (b) (10 points) Find the mutual information between X and Y.
   (c) (20 points) [For 535a students only] Show that the input distribution $\mathcal{N}(0, \sigma_X^2)$ maximizes $I(X;Y)$ for inputs with finite power ($\mathbb{E}[X^2] \leq P$).

(a) $Y = X + Z$ since X and Z are Gaussian, Y has also Gaussian dist.

$EY = EX + EZ = 0$, $Var(Y) = Var(X) + Var(Z) = \sigma_X^2 + \sigma_N^2$

So, the distribution of Y is $N(0, \sigma_X^2 + \sigma_N^2)$.

(b) $I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(X+Z|X)$

$= H(Y) - H(Z|X)$

$= H(Y) - H(Z)$

(as $Y \sim N(0, \sigma_X^2 + \sigma_N^2)$ and $Z \sim N(0, \sigma_N^2)$) $= \frac{1}{2}\log(2\pi e(\sigma_X^2 + \sigma_N^2)) - \frac{1}{2}\log(2\pi e(\sigma_N^2))$

$= \frac{1}{2}\log(1 + \frac{\sigma_X^2}{\sigma_N^2})$.

(c) [ECE 535a students]

The capacity of the Gaussian channel with the power constraint is

$$C = \max_{f(x):\, EX^2 \leq P} I(X;Y)$$

We can calculate the capacity as follows: Expanding $I(X;Y)$, we have

$$\begin{aligned} I(X;Y) &= h(Y) - h(Y|X) \\ &= h(Y) - h(X+Z|X) \\ &= h(Y) - h(Z|X) \\ &= h(Y) - h(Z). \end{aligned}$$

Now, $h(Z) = \frac{1}{2}\log(2\pi e\, \sigma_N^2)$. Also,

$Var(Y) = Var(X) + Var(Z) = P + \sigma_Z^2$. or $EY^2 = EX^2 + EZ^2 = P + \sigma_N^2$

Given $EY^2 = P+N$, the entropy of Y is bounded by $\frac{1}{2}\log(2\pi e(P+\sigma_N^2))$ by the Q3 of HW3 (the normal distribution maximizes the entropy for a given variance). Applying the result to mutual information, we obtain

$$I(X;Y) = h(Y) - h(Z) \leq \tfrac{1}{2}\log(2\pi e(P+\sigma_N^2)) - \tfrac{1}{2}\log(2\pi e\,\sigma_N^2) = \tfrac{1}{2}\log(1 + P/\sigma_N^2)$$

Hence, the capacity of the Gaussian channel is

$$C = \max_{EX^2 \leq P} I(X;Y) = \tfrac{1}{2}\log(1 + P/\sigma_N^2)$$ and the maximum is achieved when $X \sim N(0, P)$.

(So, X needs to have Gaussian distribution.)

3. Let X be a binary memoryless source producing symbols with probability $p(X=0)=p$ and $p(X=1)=1-p$. The source output is transmitted over a Gaussian channel, $Y=X+Z$, where the noise Z is drawn from a Gaussian distribution $\mathcal{N}(0,\sigma_N^2)$.

   (a) (15 points) Write an expression for $I(X;Y)$.

   (b) (15 points) Let X be a memoryless source with letters $\{x_1,x_2,\ldots,x_M\}$ and the corresponding probabilities $\{p_1,p_2,\ldots,p_M\}$. Find the mutual information between X and Y for the given Gaussian channel.

(a) In order to compute $I(X;Y)$, we need to compute $f_Y(y)$:

$$I(X;Y)=H(Y)-H(Y|X)=H(Y)-H(X+Z|X)=H(Y)-H(Z)$$

$$f_Y(y)=\frac{d}{dy}F_Y(y)$$

$$F_Y(y)=P(Y\leq y)=p(X+Z\leq y)=p(Z\leq y-X)=p\,P(Z\leq y)+(1-p)p(Z\leq y-1)$$

$$F_Y(y)=pF_Z(y)+(1-p)F_Z(y-1)\longrightarrow f_Y(y)=pf_Z(y)+(1-p)f_Z(y-1)$$

$$f_Y(y)=\frac{1}{\sqrt{2\pi\sigma_N^2}}\left(pe^{-y^2/2\sigma_N^2}+(1-p)e^{-\frac{(y-1)^2}{2\sigma_N^2}}\right)$$

Then, $h(y)=-\int f_Y(y)\log f_Y(y)\,dy=\ldots$ and $h(z)=\frac{1}{2}\log(2\pi e\sigma_N^2)$

Again, we need to obtain $f_Y(y)$

(b) $F_Y(y)=P(Y\leq y)=p(X+Z\leq y)=p(Z\leq y-X)=\sum_{i=1}^{M}p_i\,P(Z\leq y-x_i)$

$h(y)=-\int f_Y(y)\log f_Y(y)\,dy=\ldots$ where $Z\sim N(0,\sigma_Z^2)$

Then, again

$$I(X;Y)=h(y)-h(y|X)=h(y)-h(z)$$

$$=h(y)-\tfrac{1}{2}\log(2\pi e\sigma_N^2)=\ldots$$

4. (20 points) (Extra-Credit) A channel with m input and n output symbols is said to be symmetric if its channel matrix has the property that its each row $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is a permutation of another row, and each column $\mathbf{q} = (q_1, q_2, \ldots, q_m)$ is a permutation of another column. Derive the expression for the channel capacity of such a symmetric channel. (Hint: prove first that the conditional entropy $H(Y|X)$ is independent of the input probability distribution).

$$H(Y|X) = \sum_{i=1}^{m} \sum_{j=1}^{n} p(x_i)\, p(y_j|x_i) \log \frac{1}{p(y_j|x_i)}$$

$$= \sum_{i=1}^{m} p(x_i) \sum_{j=1}^{n} p(y_j|x_i) \log \frac{1}{p(y_j|x_i)}$$

$$= \sum_{i=1}^{m} p(x_i) \sum_{j=1}^{n} P_j \log \frac{1}{P_j} = \sum_{j=1}^{n} P_j \log \frac{1}{P_j}$$

We have proved that $H(Y|X)$ does not depend on the input dist. $\max_{p(X)} I(X;Y) = \max_{p(X)} H(Y)$

We know that $H(Y) \leq \log_2 n$ where the equality is achieved when $p(y_j) = \frac{1}{n}$, $1 \leq j \leq n$.

$$p(y_j) = \sum_{i=1}^{m} p(x_i) P_{ij} = \sum_{i=1}^{m} p(x_i) P_{ij} \quad \text{if } p(x_i) = \frac{1}{m}$$

$$= \frac{1}{m} \sum_{i=1}^{m} P_{ij}$$

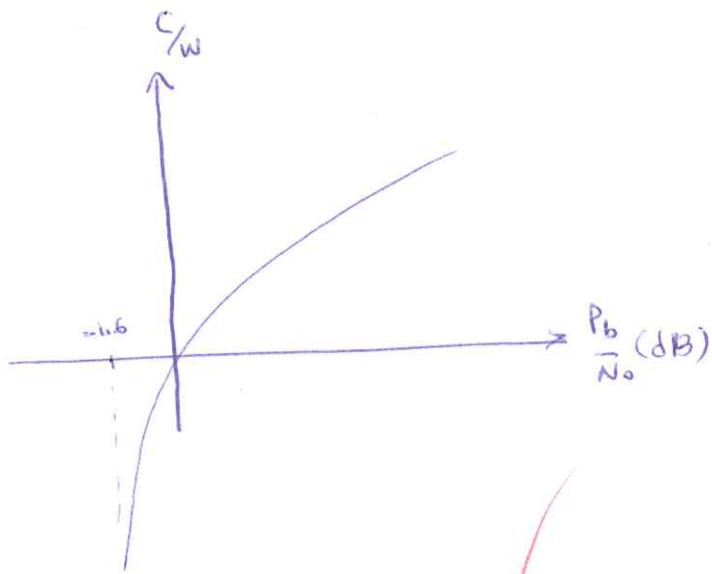
Thus all symbols $y \in Y$ have the same probability and the capacity of a symmetric channel is given by $C = \log n + \sum_{j=1}^{n} P_j \log_2 (P_j)$

5. (Extra-Credit) According to the channel capacity of the band-limited continuous channel with signal power P_b, noise variance N_0 and bandwidth W: $C = W\log(1 + \frac{P_b}{N_0 W})$, answer the following questions with justification.

(a) (5 points) Plot $\frac{C}{W}$ as a function of $\frac{P_b}{N_0}$. (Hint: $P_b = CE_b$, where E_b is the signal energy per bit.)

(b) (5 points) What is the minimum power P^* that is required to send 1 bit reliably?

(a)



(b) $\frac{C}{W} \leq \log_2\left(1 + \frac{P_b}{N_0 W}\right) = \log_2\left(1 + \frac{C}{W}\frac{E_b}{N_0}\right)$, where $P_b = CE_b$

$$\frac{E_b}{N_0} \geq \frac{2^{C/W} - 1}{\frac{C}{W}}.$$

In the case $C = 1$ bit $\rightarrow$ $\frac{E_b}{N_0} \geq \frac{2^{1/W} - 1}{1/W}$ ...