The Solutions for Quiz 3

1)

a) As an orthonormal set of basis functions we consider the set

$$\psi_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad \psi_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases}$$

$$\psi_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases} \qquad \psi_4(t) = \begin{cases} 1 & 3 \le t < 4 \\ 0 & \text{o.w} \end{cases}$$

and

$$\begin{pmatrix} s_1(t) \\ s_2(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \\ \psi_4(t) \end{pmatrix}$$

The dimensionality of the waveforms are 4.

b) The representation vectors are

$$\mathbf{s}_1 = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$$
$$\mathbf{s}_2 = \begin{bmatrix} 1 & -2 & -2 & 2 \end{bmatrix}$$

c) The distance between the first and the second vector is

$$d_{1,2} = \sqrt{|\mathbf{s}_3 - \mathbf{s}_4|^2} = \sqrt{\left| \begin{bmatrix} 0 & 1 & 3 & -3 \end{bmatrix} \right|^2} = \sqrt{19}$$

2)

a) The expansion coefficients $\{c_n\}$, that minimize the mean square error, satisfy

$$c_n = \int_{-\infty}^{\infty} x(t)\psi_n(t)dt = \int_0^4 \sin\frac{\pi t}{4}\psi_n(t)dt$$

Hence,

$$\begin{aligned} c_1 &= \int_0^4 \sin\frac{\pi t}{4}\psi_1(t)dt = \frac{1}{2}\int_0^2 \sin\frac{\pi t}{4}dt - \frac{1}{2}\int_2^4 \sin\frac{\pi t}{4}dt \\ &= -\frac{2}{\pi}\cos\frac{\pi t}{4}\Big|_0^2 + \frac{2}{\pi}\cos\frac{\pi t}{4}\Big|_2^4 \\ &= -\frac{2}{\pi}(0-1) + \frac{2}{\pi}(-1-0) = 0 \end{aligned}$$

Similarly,

$$\begin{aligned} c_2 &= \int_0^4 \sin\frac{\pi t}{4}\psi_2(t)dt = \frac{1}{2}\int_0^4 \sin\frac{\pi t}{4}dt \\ &= -\frac{2}{\pi}\cos\frac{\pi t}{4}\Big|_0^4 = -\frac{2}{\pi}(-1-1) = \frac{4}{\pi} \end{aligned}$$

and

$$\begin{aligned} c_3 &= \int_0^4 \sin\frac{\pi t}{4}\psi_3(t)dt \\ &= \frac{1}{2}\int_0^1 \sin\frac{\pi t}{4}dt - \frac{1}{2}\int_1^2 \sin\frac{\pi t}{4}dt + \frac{1}{2}\int_2^3 \sin\frac{\pi t}{4}dt - \frac{1}{2}\int_3^4 \sin\frac{\pi t}{4}dt \\ &= 0 \end{aligned}$$

Note that c_1, c_2 can be found by inspection since $\sin\frac{\pi t}{4}$ is even with respect to the $x = 2$ axis and $\psi_1(t)$, $\psi_3(t)$ are odd with respect to the same axis.

b) The residual mean square error $E_{\min}$ can be found from

$$E_{\min} = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{3} |c_i|^2$$

Thus,

$$\begin{aligned} E_{\min} &= \int_0^4 \left(\sin\frac{\pi t}{4}\right)^2 dt - \left(\frac{4}{\pi}\right)^2 = \frac{1}{2}\int_0^4 \left(1 - \cos\frac{\pi t}{2}\right) dt - \frac{16}{\pi^2} \\ &= 2 - \frac{1}{\pi}\sin\frac{\pi t}{2}\Big|_0^4 - \frac{16}{\pi^2} = 2 - \frac{16}{\pi^2} \end{aligned}$$