Quiz I - Solution:

(Q 1) $H(X,Y) \leq H(X) + H(Y)$

First, we show that $H(X,Y) = H(X) + H(Y|X)$. For this, we have:

$$H(X,Y) = -\sum_{x,y} p(x,y) \log p(x,y) = -\sum_{x,y} p(x,y) \log p(x)\,p(y|x)$$

$$= -\sum_{x,y} p(x,y) \log p(x) - \sum_{x,y} p(x,y) \log p(y|x)$$

$$= -\sum_{x} p(x) \log p(x) - \sum_{x,y} p(x,y) \log p(y|x)$$

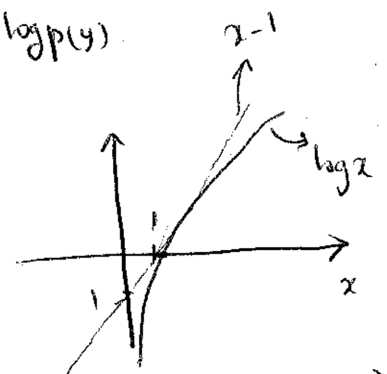
$$= H(X) + H(Y|X).$$

The second is that to show $H(Y|X) \leq H(Y)$. Then, we are done.

For this, we have:

$$H(Y|X) - H(Y) = -\sum_{x,y} p(x,y) \log p(y|x) - \sum_{x,y} p(x,y) \log p(y)$$

$$= +\sum_{x,y} p(x,y) \log \frac{p(y)}{p(y|x)}$$



$$\leq +\sum_{x,y} p(x,y) \left(\frac{p(y)}{p(y|x)} - 1 \right) \qquad (\log_2 x \leq x-1)$$

$$= \sum_{x,y} p(x,y) \frac{p(y)\,p(x)}{p(x,y)} - \sum_{x,y} p(x,y) = \underbrace{\sum_{x,y} p(y)\,p(x)}_{\sum_y p(y) \sum_x p(x)} - 1 = 0$$

(Q1) ...

Therefore, $H(Y) - H(Y|X) \geq 0$ which implies that

$$H(Y) \geq H(Y|X).$$

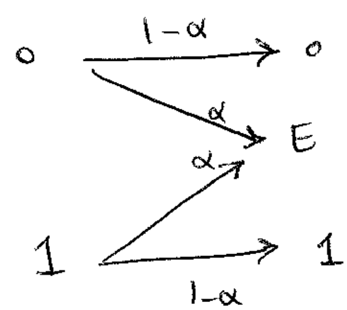
According to the chain rule (what we also proved it),

$$H(X,Y) = H(X) + H(Y|X)$$

$$\overset{(a)}{\leq} H(X) + H(Y)$$

where (a) comes from our claim $(H(Y) \geq H(Y|X))$ which we have also proved it. ∎

(Q2) Capacity of Binary Erasure Channel (α).



- We calculate the capacity of the binary erasure channel as follows:

$$C = \max_{p(x)} I(X;Y)$$

$$C = \max_{p(x)} H(Y) - H(Y|X)$$

Then, $H(Y|X) = -\sum_{x,y} p(x,y) \log p(y|x) = \cdots = -\alpha \log \alpha - (1-\alpha) \log (1-\alpha)$

let $p(X=1) = \pi$, we have for $H(Y)$:

$H(Y)$:

$$p(y=0) = p(y=0|x=0)\,p(x=0) + \overbrace{p(y=0|x=1)}^{0}\,p(x=1)$$

$$p(y=0) = (1-\alpha)(1-\pi)$$

$$p(y=1) = p(y=1|x=0)\,p(x=0) + p(y=1|x=1)\,p(x=1)$$

$$p(y=1) = (1-\alpha)\pi$$

$$p(y=E) = p(y=E|x=0)\,p(x=0) + p(y=E|x=1)\,p(x=1)$$

$$p(y=E) = \alpha$$

$$H(Y) = -\sum_{y} p(y)\log p(y) = \dots = H(\alpha) + (1-\alpha)H(\pi)$$

, where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$.

So, $C = \max_{p(x)} \{H(Y) - H(Y|X)\} = \max_{\pi} (1-\alpha)H(\pi) + H(\alpha) - H(\alpha)$

$= \max_{\pi} (1-\alpha)H(\pi) = (1-\alpha)\max_{\pi} H(\pi)$

$= (1-\alpha)$ where X has uniform distributio over {0,1}