Error floor

Frame Error Rate (FER) on BSC

$$FER(\alpha) = \sum_{k=i}^{n} c_k \alpha^k (1-\alpha)^{(n-k)}$$

- c_k - the number of configurations of received bits for which k channel error lead to a codeword (frame) error
- i - the minimal number of channel errors that can lead to a decoding error
- When $\alpha \ll 1$

$$\log(FER(\alpha)) \approx \log(c_i) + i\log(\alpha)$$

Frame Error Rate (FER)

- What is usually plotted (semi-log scale):

$$\log(FER(\alpha)) = \log\Big(\sum_{k=i}^{n} c_k \alpha^k (1-\alpha)^{n-k}\Big)$$

$$= \log(c_i) + i\log(\alpha) + \log((1-\alpha)^{n-i})$$

$$+\log\left(1 + \frac{c_{i+1}}{c_i}\alpha(1-\alpha)^{-1} + \ldots + \frac{c_n}{c_i}\alpha^{n-i}(1-\alpha)^{i-n}\right)$$

- As error probability decreases…

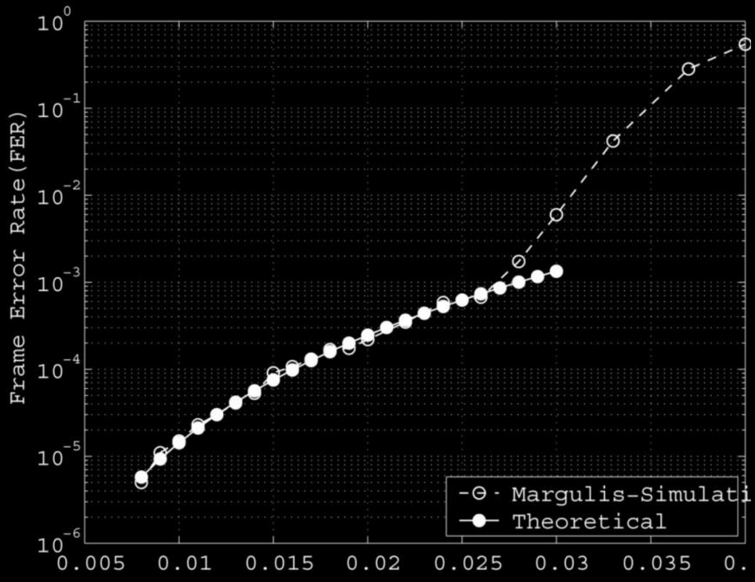
$$\lim_{\alpha\to 0}\left[\log((1-\alpha)^{n-i})\right] = 0$$

$$\lim_{\alpha\to 0}\left[\log\left(1+\frac{c_{i+1}}{c_i}\alpha(1-\alpha)^{-1}\ldots+\frac{c_n}{c_i}\alpha^{n-i}(1-\alpha)^{i-n}\right)\right] = 0$$

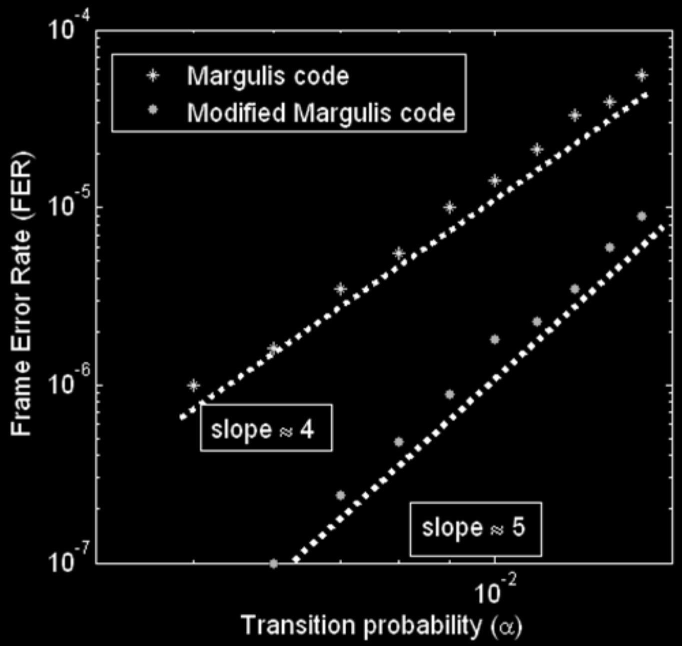
$$\log(FER(\alpha)) \approx \log(c_i) + i\log(\alpha)$$

Practical problems related to error floor

Error floor estimation



Code construction



Trapping sets

Basic terminology

- An *eventually correct* variable node
- *A fixed point* of iterative decoding
- *Inducing set*
- *Fixed set*
- The *critical number* m of a trapping set is the minimal number of variable nodes that have to be initially in error for the decoder to end up in that trapping set

Basic terminology

- Consider an LDPC code of length n, and assume that the all-zero codeword is transmitted over the BSC, and that the word $\mathbf{y}$ is received.
- Let $\mathbf{x}^l$, $l \leq D$ be the decoder output vector at the l th iteration (D the maximum number of iterations).
- A variable node v is said to be *eventually correct* if there exists a positive integer q such that for all $l \geq q$,

$$v \notin \operatorname{supp}(\mathbf{x}^l)$$

- A *decoder failure* is said to have occurred if there does not exist $l \leq D$ such that

$$\operatorname{supp}(\mathbf{x}^l) = \emptyset.$$

Definitions

- *Definition 1: Let $T(\mathbf{y})$ denote the set of variable nodes that* are not eventually correct. If $T(\mathbf{y}) \neq \emptyset$, let $a = |T(\mathbf{y})|$ and b be the number of odd degree check nodes in the sub-graph induced by $T(\mathbf{y})$. We say $T(\mathbf{y})$ is an (a, b) trapping set.
- Note that for each failure of the iterative decoder, there is a corresponding set of corrupt variable nodes

$$F = \text{supp}(\mathbf{x}^D)$$

- The set F is not necessarily a trapping set because it may not contain all the variable nodes that are eventually incorrect, such as variable nodes that oscillate between the right value and the wrong value.

Inducing sets and fixed sets

- *Definition 2:* Let T be a trapping set. If $\mathbf{T}(\mathbf{y}) = T$ then $\mathrm{supp}(\mathbf{y})$ is an ***inducing set*** of T.
- *Definition 3: Let* T be a trapping set and let $\mathbf{Y}(T) = \{\mathbf{y} \,|\, \mathbf{T}(\mathbf{y}) = T\}$. The ***critical number*** $m(T)$ of trapping set T is the minimal number of variable nodes that have to be initially in error for the decoder to end up in the trapping set T , i.e. $m(T) = \min_{Y(T)} |\mathrm{supp}(\mathbf{y})|$
- *Definition 4:* The vector $\mathbf{y}$ is *a* ***fixed point*** of the decoding algorithm if $\mathrm{supp}(\mathbf{y}) = \mathrm{supp}(\mathbf{x}^l)$ for all l.
- *Definition 5:* If $T(\mathbf{y})$ is a trapping set and $\mathbf{y}$ is a fixed point, then $T(\mathbf{y}) = \mathrm{supp}(\mathbf{y})$ is called a ***fixed set***.

Trapping sets for column weight-three codes

- *Theorem* [Chillapagari *et al.*, (2009)]: (sufficient conditions) Let Γ be a subgraph induced by the set of variable nodes T. Let the checks in Γ can be partitioned into two disjoint subsets: E consisting of checks with even degree, and O consisting of checks with odd degree. The vector $\boldsymbol{y}$ is a fixed set if :

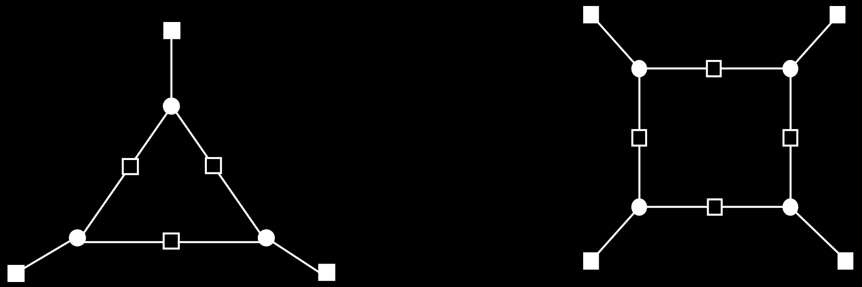
  (a) $\mathrm{supp}(\mathbf{y})=T$,

  (b) Every variable node in Γ is connected to at least two checks in E ,

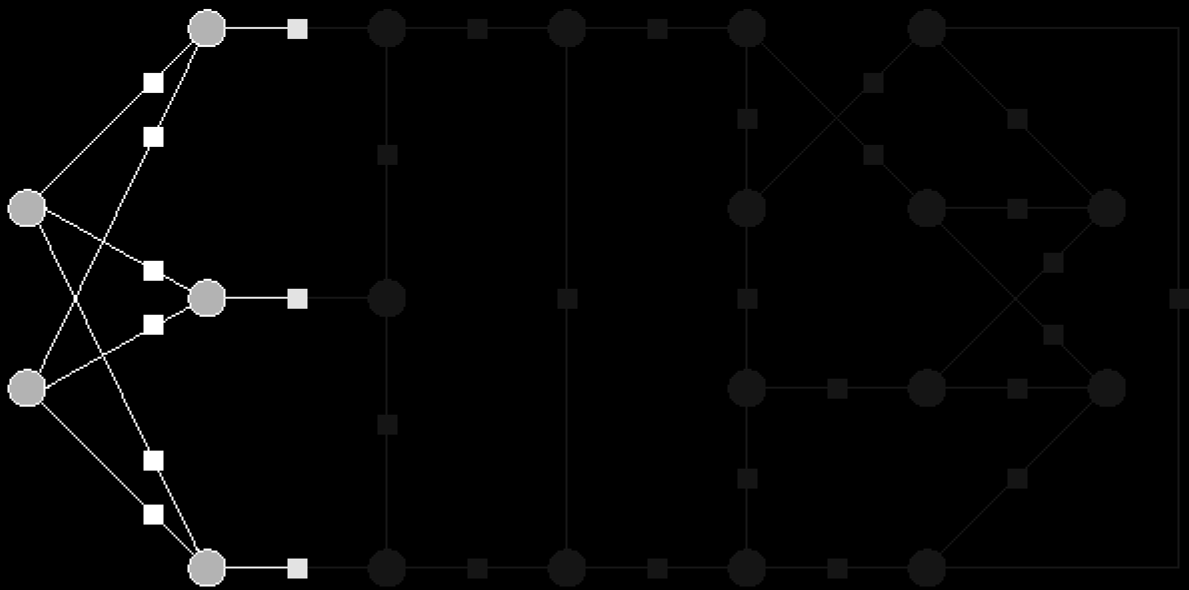
  (c) No two checks of O are connected to a variable node outside Γ.

The (a,b) notation

- <u>An (a,b) trapping set</u>: a set of not eventually correct variabe nodes of size a, and the b odd degree check nodes in the sub-graph induced by these variable nodes.

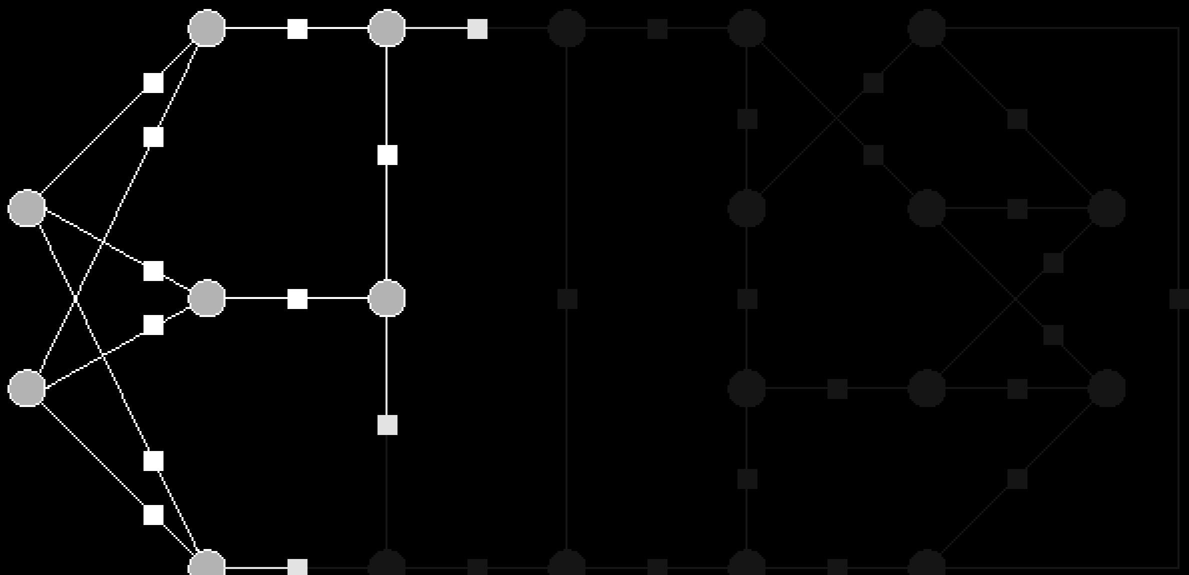


Bigger trapping sets from smaller (1)



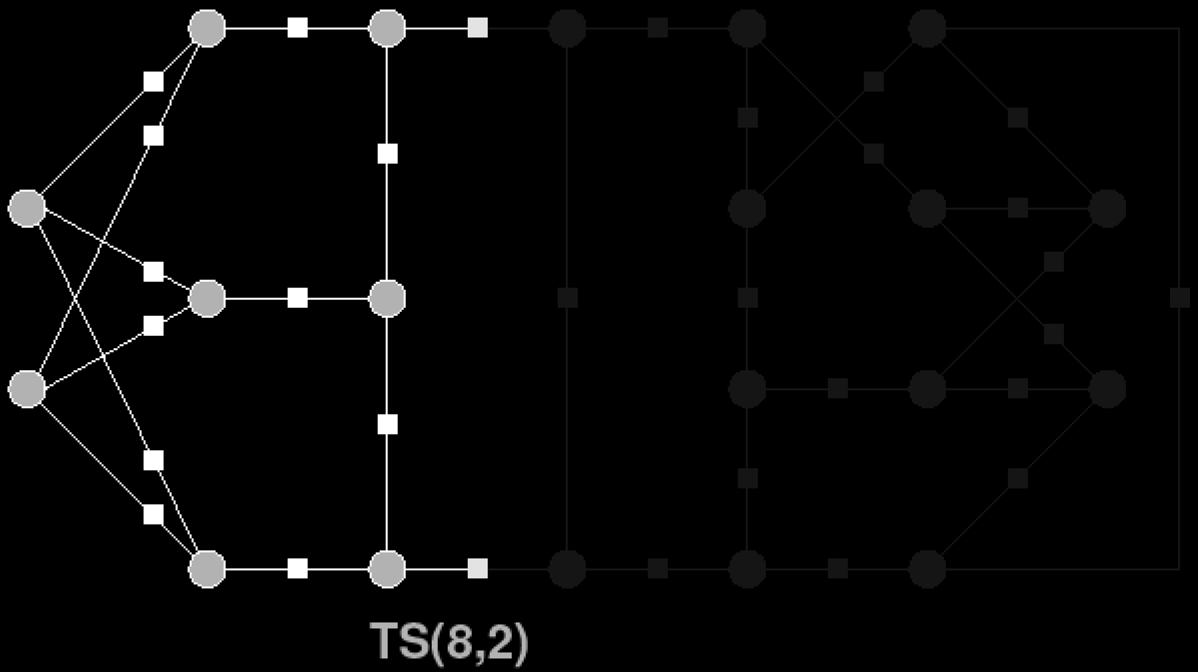
TS(5,3)

Bigger trapping sets from smaller (2)

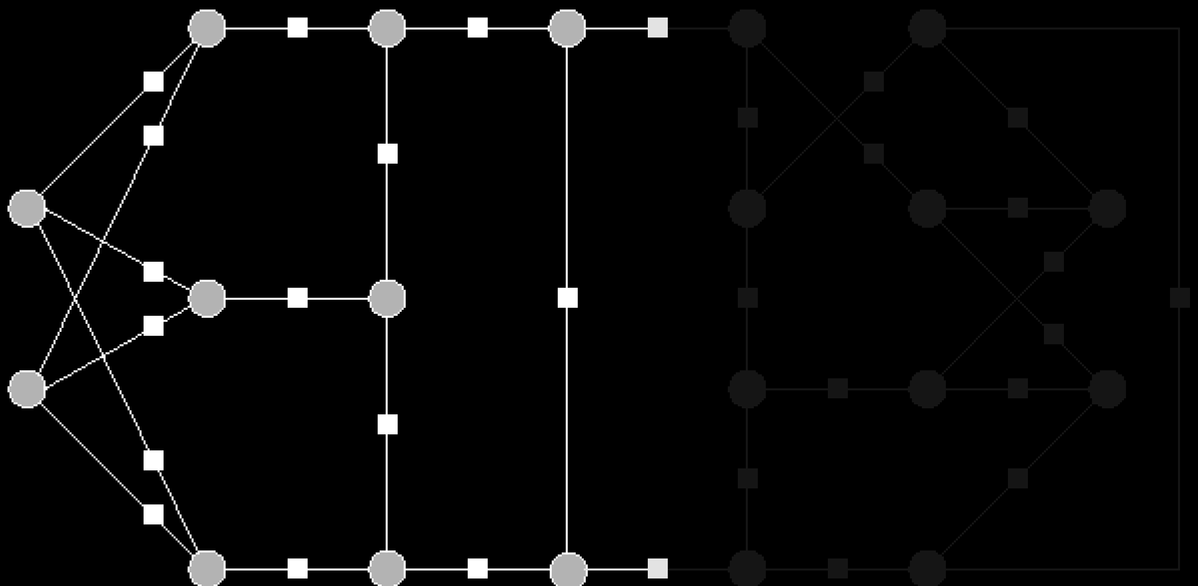


TS(7,3)

Bigger trapping sets from smaller (3)

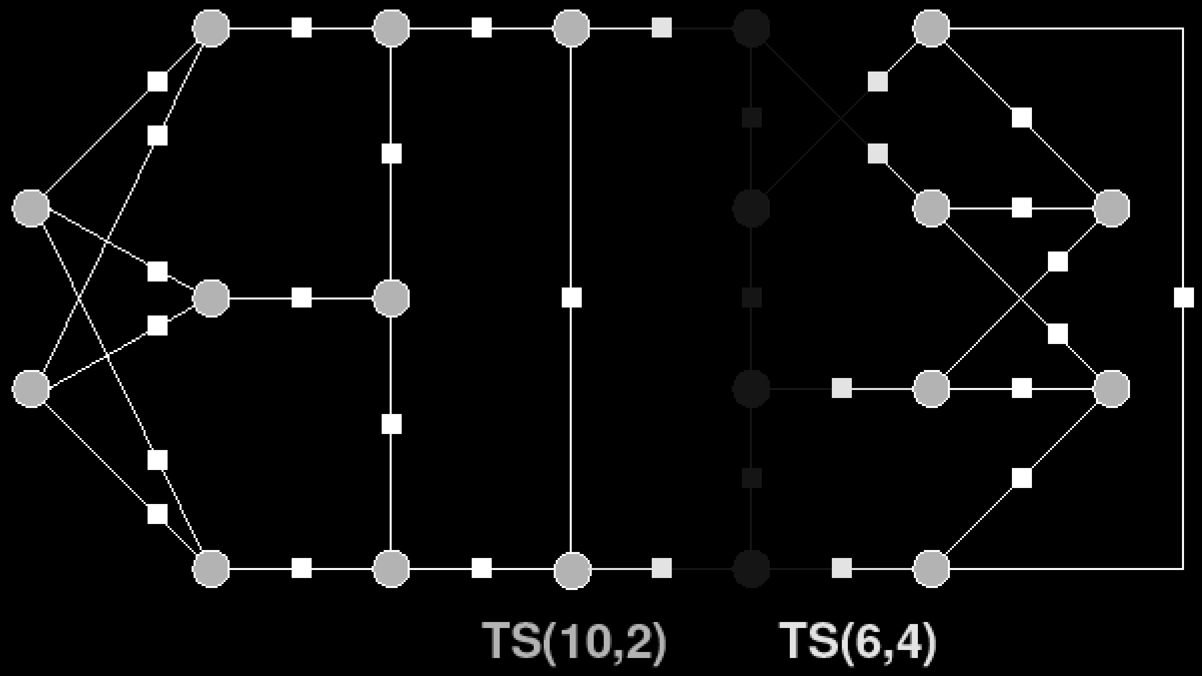


Bigger trapping sets from smaller (4)

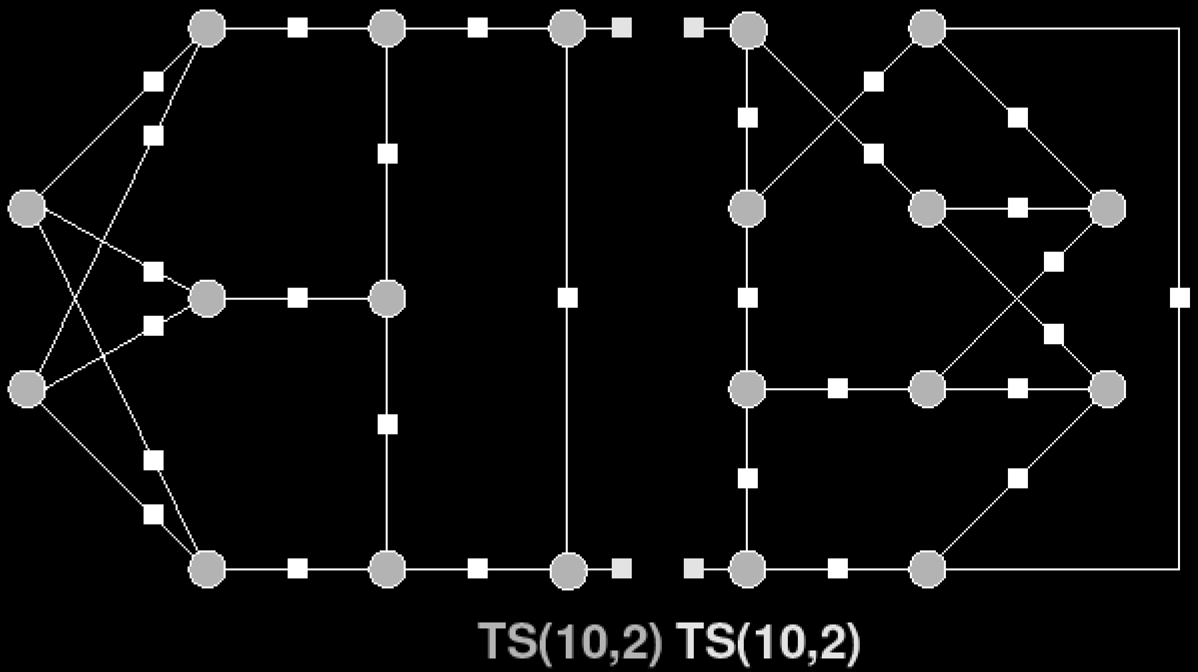


TS(10,2)

Bigger trapping sets from smaller (5)



Bigger trapping sets from smaller (6)

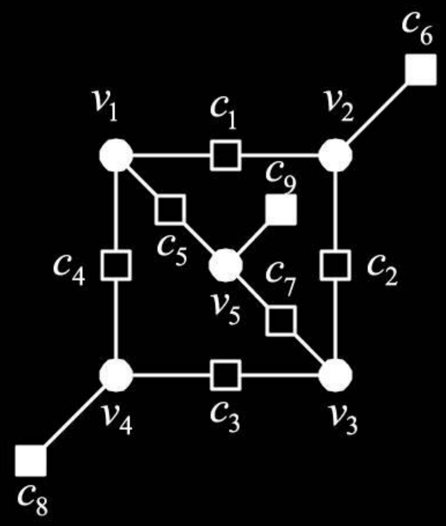


Trapping set as decoding failures

- The all zero codeword is transmitted.
- The decoder performs *D* iterations.
  - $\boldsymbol{y} = (y_1\ y_2\ .\ .\ .\ y_n)$ - decoder input
  - $\boldsymbol{x}^l$, $l \leq D$ -the decoder output vector at the *l*-th iteration
- A variable node *v* is *eventually correct if there exists a positive integer* d *such that for all* $l > d$, $v \notin \text{supp}(\boldsymbol{x}^l)$.
- *A decoder failure is said to occur if there* does not exits $l \leq D$ such that $\text{supp}(x^l) = \emptyset$.
  - *T(**y**)* – a nonempty set of variable nodes that are not eventually correct
  - G - subgraph induced by *T(**y**), C(*G) =$E \cup O$ (even and odd degree check nodes in)
  - T(***y***) is an (*a,b*) trapping set, where a = |T(**y**)|, *b* =|O|

Trapping set harmfulness

- Example BSC:
- Error patterns leading to a decoding failure
- Bit flipping algorithm: $\{v_1, v_3\}, \{v_2, v_4\}, \{v_1, v_2, v_3\}\ldots$
- Gallager A/B algorithm: $\{v_2, v_4, v_5\}$
- LP decoder: $\{v_1, v_2, v_3, v_4, v_5\}$



Critical number

- With every trapping set T is associated a *critical number* m (or $m(T)$) defined as the minimum number of nodes in T that have to be initially in error for the decoder to end in that trapping set.
- Smaller values of m mean that fewer number of errors can result in decoding failure by ending in that trapping set.

Strength of a trapping set

- Not all configurations of m errors in a trapping set result in a decoding failure.
  - (5, 3) TS: m=3, only one configuration of three errors leads to a decoding failure.

- A set of m erroneous variable nodes which leads to a decoding failure by ending in a trapping set T of class X is called a *failure set* of X .
- The number of failure sets of T is called the *strength of* T and is denoted by s. A class X has $s|X|$ failure sets.

More ambitious goal

- The decoding failures for various algorithms on different channels are closely related and are dependent on only a few topological structures.
- These structures are either trapping sets for iterative decoding algorithms on the BSC or larger subgraphs containing these trapping sets.
- On the BSC, trapping sets are subgraphs formed by cycles or union of cycles.
- Ultimate goal: *Find topological interrelations among trapping sets/topological interrelations among error patterns that cause decoding failures for various algorithms on different channels.*