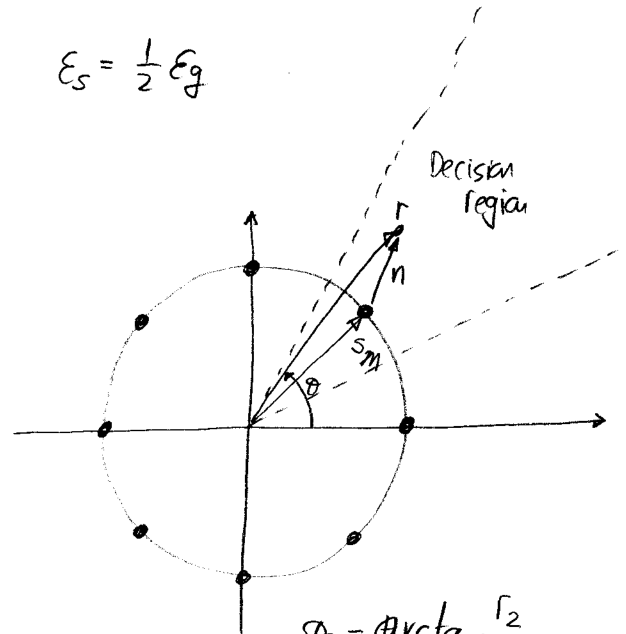
Probability of Error for M-ary PSK

$$s_m(t) = g(t)\cos\left(2\pi f_c t + 2\pi \frac{m-1}{M}\right) \qquad 1 \le m \le M, \quad 0 \le t < T$$

$$\bar{s}_m = \left(\sqrt{\mathcal{E}_s}\cos\left(2\pi\frac{m-1}{M}\right), \sqrt{\mathcal{E}_s}\sin\left(2\pi\frac{m-1}{M}\right)\right), \qquad \mathcal{E}_s = \frac{1}{2}\mathcal{E}_g$$



- The signal waveforms have equal energy
- Optimum AWGN detector computes

$$C(\bar{r}, \bar{s}_m) = \bar{r} \cdot \bar{s}_m \qquad 1 \le m \le M$$

$$\theta = \operatorname{arctg}\frac{r_2}{r_1}$$

Assume that s_1 was transmitted

$$s_1 = (\sqrt{\mathcal{E}_s}, 0) \quad \text{- phase zero}$$

$$r_1 = \sqrt{\mathcal{E}_s} + n_1$$
$$r_2 = n_2$$

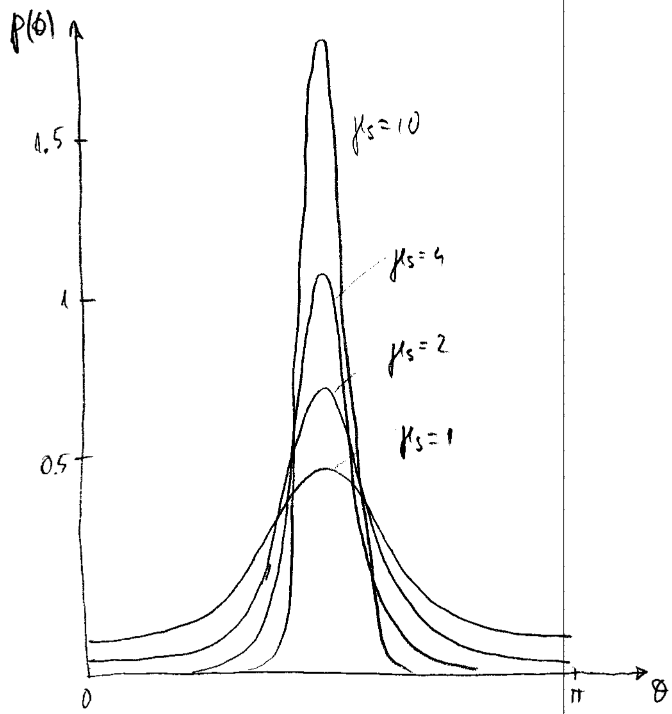
$$p(r_1, r_2) = p(r_1)\,p(r_2) = \frac{1}{2\pi\sigma_r^2} e^{-\frac{(r_1 - \sqrt{\mathcal{E}_s})^2 + r_2^2}{2\sigma_r^2}} \qquad n_1, n_2 : N(0, \sigma_r^2)$$

$$R^2 = r_1^2 + r_2^2, \quad tg\,\theta = \frac{r_2}{r_1} \qquad J = \begin{vmatrix} \frac{\partial r_1}{\partial R} & \frac{\partial r_1}{\partial \theta} \\ \frac{\partial r_2}{\partial R} & \frac{\partial r_2}{\partial \theta} \end{vmatrix} = R$$

$$p(R, \theta) = \frac{R}{2\pi\sigma_r^2} e^{-\frac{R^2 + \mathcal{E}_s - 2R\sqrt{\mathcal{E}_s}\cos\theta_r}{2\sigma_r^2}} \qquad p(R,\theta) = |J| \cdot p(r_1, r_2)$$

$$p(\theta) = \int_0^{+\infty} p(R,\theta)\,dR = \frac{1}{2\pi} e^{-\gamma_s \sin^2\theta_r} \cdot \int_0^{+\infty} R\, e^{-(R - \sqrt{2\gamma_s}\cos\theta_r)^2/2}\, dR$$

$$\gamma_s = \frac{\mathcal{E}_s}{N_0}$$



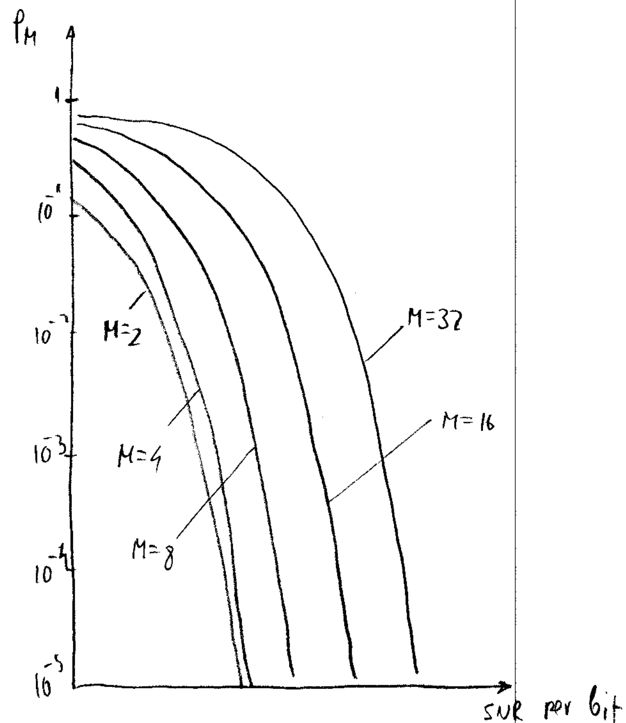
$$P_M = 1 - \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} p(\theta)\, d\theta \qquad\qquad P_M = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

M=2 $\quad P_2 = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$

M=4
- two binary phase-modulated signals in phase quadrature.
- there is no crosstalk between the signals on the two quadrature carriers

$$P_c = (1-P_2)^2 = \left(1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right)^2$$

$$P_4 = 1 - P_c = 1 - (1-P_2)^2 = P_2(2-P_2) = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\left(2 - Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)\right)$$



- We assume that the demodulator has a perfect estimate of carrier phase
- In practice, carrier phase is extracted from the received signal by some nonlinear operation that introduces phase ambiguity
  - In binary PSK, signal is squared, frequency is doubled
  - 4-PSK - fourth harmonic is taken
- Phase ambiguity problem can be solved by differentialy encoded phases →

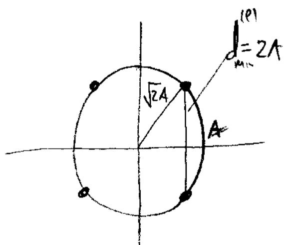
Probability of Error for QAM

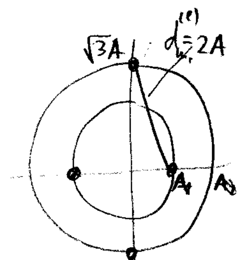
$$s_m(t) = A_{m,c}\, g(t) \cos w_c t - A_{m,s}\, g(t) \sin w_c t \qquad 0 \le t < T \qquad 1 \le m \le M$$

$A_{m,c}$, $A_{m,s}$ – information-bearing amplitudes of quadrature carriers
$g(t)$ – pulse shape

$$\bar{s}_m = \left(A_{m,c}\sqrt{\frac{\mathcal{E}_g}{2}},\ A_{m,s}\sqrt{\frac{\mathcal{E}_g}{2}} \right)$$

Start with $M = 4$:





$P_{av} = \frac{1}{4} \cdot 4 \cdot 2A^2$

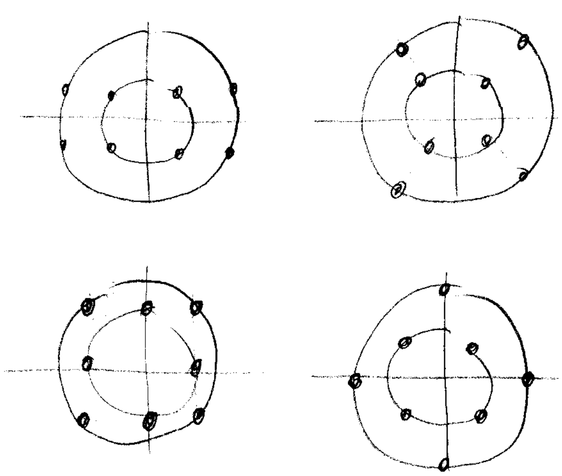
$P_{av} = \frac{1}{2}\left(2(3A^2) + 2A^2 \right) = 4A^2$

– Two constelations:
1) – equal amplitudes
2) – different amplitudes

– same P_{av}
– same $d^{(e)}_{min}$

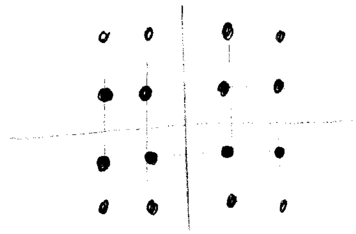
$\Rightarrow$ no P_e difference

$M = 8$



the best one – compute P_{av}
– $d^{(e)}_{min}$

M-ary QAM with Rectangular Constellations



$\Leftrightarrow$ two PAM signals on quadrature carriers each having $\sqrt{M}$ signaling points

$$P_c = \left(1 - P_{\sqrt{M}}\right)^2$$

— both PAM decisions must be correct

PAM: $$P_{\sqrt{M}} = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3}{M-1} \frac{\mathcal{E}_{av}}{N_0}}\right)$$

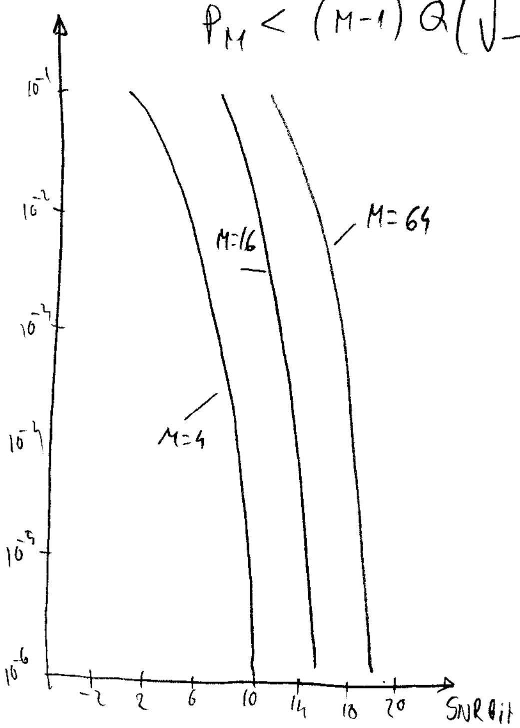
$$P_M = 1 - P_c = 1 - \left(1 - P_{\sqrt{M}}\right)^2$$ — probability of symbol error

$$P_M \leq 1 - \left(1 - 2Q\left(\sqrt{\frac{3\mathcal{E}_{av}}{(M-1)N_0}}\right)\right) \leq 4Q\left(\sqrt{\frac{3k\mathcal{E}_{bav}}{(M-1)N_0}}\right)$$

because for PAM $$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6 \log_2 M \, \mathcal{E}_{bav}}{(M^2-1)N_0}}\right)$$

For nonrectangular: use Union bound

$$P_M < (M-1)\, Q\left(\sqrt{\frac{d_{min}^{(e)\,2}}{2N_0}}\right)$$



Comparison between M-ary PSK and QAM

$$P_{\sqrt{M}} = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\,E_{av}}{(M-1)N_0}}\right) \quad \text{— QAM}$$

$$P_M \cong 2Q\left(\sqrt{2\gamma_s} \cdot \sin\frac{\pi}{M}\right) \quad \text{— M-ary PSK}$$

SNR gain of M-QAM over M-ary PSK

M	gain (dB)
8	1.65
16	4.20
32	7.02
64	9.95