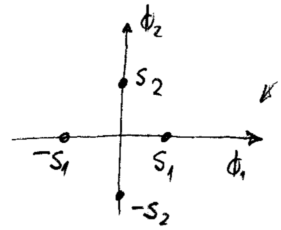
Probability of Error for M-ary Biorthogonal Signals

- Start with $M/2$ orthogonal signals
- Include negatives



Example

- Assume $\bar{s}_1$ was sent $(+\bar{s}_1)$
- The received vector is

$$\bar{r} = (\sqrt{\mathcal{E}_s} + n_1, n_2, \ldots, n_{\frac{M}{2}}) \qquad n_m - N(0, \tfrac{1}{2}N_0) \qquad 1 \le m \le \tfrac{M}{2}$$

- Optimum detector decides in favor of the signal producing the largest crosscorrelation

$$C(\bar{r}, \bar{s}_m) = \bar{r} \cdot \bar{s}_m = \sum_{k=1}^{M/2} r_k \cdot s_{m,k} \qquad 1 \le m \le \frac{M}{2}$$

- The sign of the largest term is used to decide whether s_m or $-s_m$ was transmitted
- Detector makes a correct decision when:
  1) $r_1 > 0$
  2) $r_1 > |r_m| \qquad 2 \le m \le \frac{M}{2}$

$$P_r\{r_1 > |r_m| \mid r_1 > 0\} = Pr\{r_1 > |n_m| \mid r_1 > 0\} = \frac{1}{\sqrt{\pi N_0}} \int_{-r_1}^{r_1} e^{-\frac{n_m^2}{N_0}} dn_m$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\frac{r_1}{\sqrt{N_0/2}}}^{\frac{r_1}{\sqrt{N_0/2}}} e^{-\frac{x^2}{2}} dx$$

$$P_c = \int_0^{+\infty} \prod_{m=2}^{\frac{M}{2}} P_r\{r_1 > |r_m| \mid r_1 > 0\}\, p(r_1)\, dr_1 \qquad \leftarrow \text{independent noise samples}$$

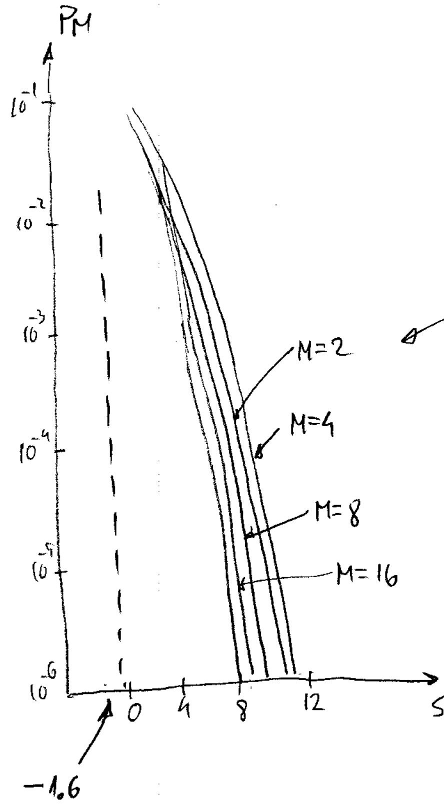
$$P_c = \int_0^{+\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\frac{r_1}{\sqrt{N_0/2}}}^{\frac{r_1}{\sqrt{N_0/2}}} e^{-\frac{x^2}{2}} dx \right)^{\frac{M}{2}-1} p(r_1)\, dr_1$$

$$P_c = \frac{1}{\sqrt{\pi N_0}} \int_0^{+\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-r_1/\sqrt{N_0/2}}^{r_1/\sqrt{N_0/2}} e^{-x^2/2}\, dx \right)^{\frac{M}{2}-1} e^{-\frac{(r_1-\sqrt{\mathcal{E}_s})^2}{N_0}}\, dr_1$$

$$P_c = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{\frac{2\mathcal{E}_s}{N_0}}}^{+\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-(v+\sqrt{2\mathcal{E}_s/N_0})}^{v+\sqrt{2\mathcal{E}_s/N_0}} e^{-x^2/2}\, dx \right)^{\frac{M}{2}-1} \cdot e^{-\frac{v^2}{2}}\, dv$$

$\mathcal{E}_s = k\mathcal{E}_b$ $\qquad 2^k = \frac{M}{2}\cdot 2 = M$

$P_M = 1 - P_c$



$M \to \infty$ → capacity limit

Because it is a symbol probability of error

$P_B \cong \frac{P_M}{2^k - 1} 2^{k-1}$

Probability of Error for M-ary Binary-Coded Signals

$\overline{C}_m = (c_{m,1}, c_{m,2}, \dots, c_{m,N})$ $\quad 1 \le m \le M$ $\quad$ – codewords

$c_{m,j} = 1 \Rightarrow s_{m,j} = +\sqrt{\frac{2\mathcal{E}_c}{T_c}} \cos 2\pi f_c t$

$c_{m,j} = 0 \Rightarrow s_{m,j} = -\sqrt{\frac{2\mathcal{E}_c}{T_c}} \cos 2\pi f_c t$

block length = dimension

$T_c = T/N, \quad \mathcal{E}_c = \mathcal{E}/N$

- A codeword is mapped to one of M waveforms $s_m(t)$
- Signal space representation

$$\overline{S}_m = (s_{m,1}, s_{m,2}, \dots, s_{m,N}) \qquad s_{m,j} = \pm\sqrt{\mathcal{E}/N} \qquad 1 \le m \le M,$$

$$P_M < (M-1) P_b = (M-1)\, Q\left(\sqrt{\frac{d_{min}^{(e)\,2}}{2N_0}}\right)$$

$$< 2^k e^{-\frac{d_{min}^{(e)\,2}}{4N_0}}$$

$d_{min}^{(e)}$ – depend on the code

Probability of Error for M-ary PAM

$$s_m(t) = A_m\, g(t) \cos 2\pi f_c t \qquad 1 \le m \le M \qquad 0 \le t < T$$

$$A_m = (2m-1-M)\, d$$

$$T = K \cdot T_b = \frac{K}{R} \qquad \mathcal{E}_m = \frac{1}{2} A_m^2 \cdot \mathcal{E}_g \qquad \mathcal{E}_g \text{ - energy of } g(t)$$

Signal space representation: $s_m = A_m \sqrt{\frac{\mathcal{E}_g}{2}}$ $\quad 1 \le m \le M$ — one dimensional

$$d_{min}^{(e)} = d\sqrt{2\mathcal{E}_g}$$

Assuming equally probable signals, the average energy is

$$\mathcal{E}_{av} = \frac{1}{M}\sum_{m=1}^{M} \mathcal{E}_m = \frac{1}{2M}\mathcal{E}_g \cdot \sum_{m=1}^{M} A_m^2 = \frac{1}{2M}\mathcal{E}_g \sum_{m=1}^{M} (2m-1-M)^2 \cdot d$$

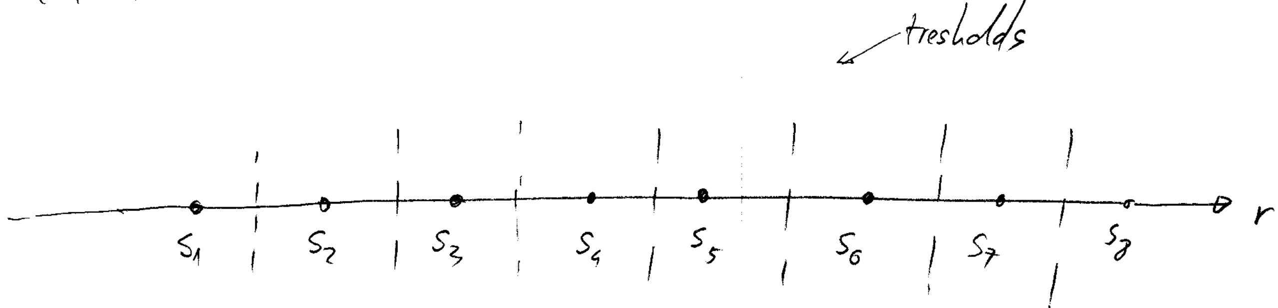
$$= \frac{d^2\mathcal{E}_g}{2M} \cdot \frac{1}{3} M (M^2-1) = \frac{1}{6}(M^2-1) d^2 \mathcal{E}_g$$

$$\left(\text{use } \sum_{m=1}^{M} (2m-1)^2 = \frac{M(2M^2-1)}{3} \right.$$

$$\left. \text{and } \sum_{m=1}^{M} (2m-1) = M^2 \right)$$

$$P_{av} = \frac{\mathcal{E}_{av}}{T} = \frac{1}{6}(M^2-1)\frac{d^2\mathcal{E}_g}{T}$$

$$C(\bar{r}, \bar{s}_m) = 2 \cdot \bar{r} \cdot \bar{s}_m - \|s_m\|^2$$



Since $d^2 \mathcal{E}_g = \dfrac{6 T P_{av}}{M^2-1}$

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6 P_{av} T}{(M^2-1) N_0}}\right)$$

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6 \mathcal{E}_{av}}{(M^2-1) N_0}}\right)$$

Express in terms of bit SNR

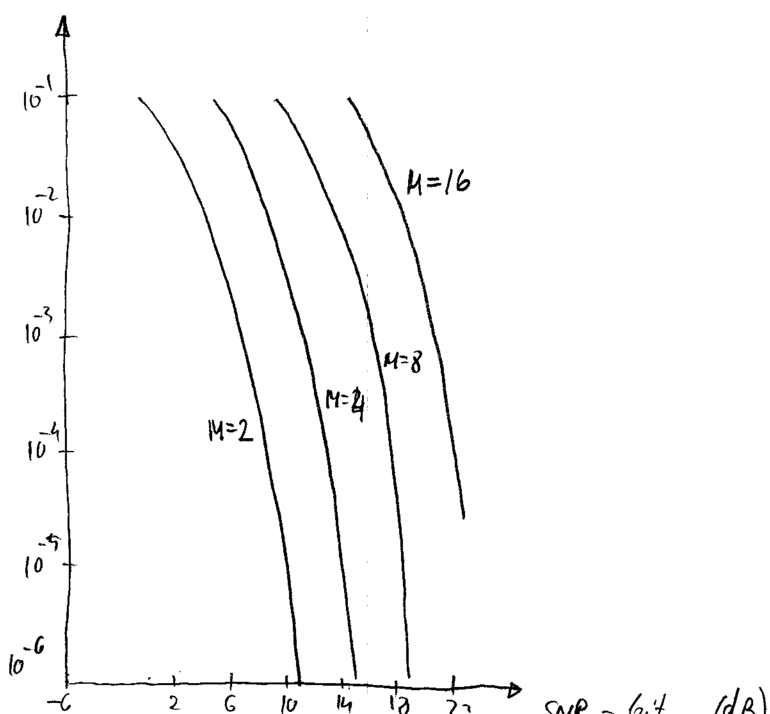
Since $T = K T_b$ $\quad 2^K = M$ $\quad \mathcal{E}_{av} = T \cdot P_{av}$

$\mathcal{E}_{b,av} = T_b \cdot P_{av}$

$$\frac{\mathcal{E}_{b,av}}{\mathcal{E}_{av}} = \frac{T}{T_b} = K = \log_2 M$$

$$P_M = \frac{2(M-1)}{M} \cdot Q\left(\sqrt{\frac{6 \log_2 M \cdot \mathcal{E}_{bav}}{(M^2-1) N_0}}\right)$$

bit SNR

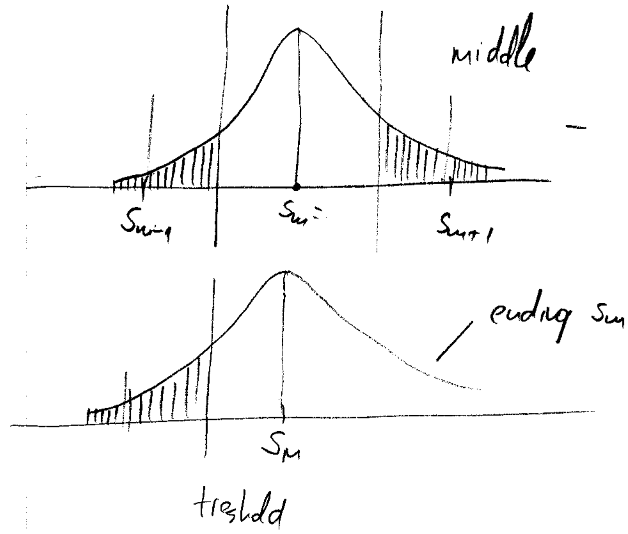


fix P_M — increasing M twice requires 4x increase of SNR

If the m-th amplitude level was transmitted:

$$r = s_m + n = \sqrt{\tfrac{1}{2}\mathcal{E}_g}\, A_m + n$$

$$n : \mathcal{N}(0, \tfrac{1}{2} N_0)$$



symbol error probability

$$P_M = \sum_{m=1}^{M} P(e|s_m) \cdot P_m = \frac{1}{M} \sum_{m=1}^{M} P(e|s_m)$$

$$= \frac{1}{M} P(e|s_1) + \frac{1}{M}\sum_{m=2}^{M-1} P(e|s_m) + \frac{1}{M} P(e|s_M)$$

endpoint   endpoint

$$\left.\begin{aligned} P(e|s_1) &= \Pr\left\{ n > d\sqrt{\tfrac{\mathcal{E}_g}{2}} \right\} \\ P(e|s_M) &= \Pr\left\{ n < -d\sqrt{\tfrac{\mathcal{E}_g}{2}} \right\} \end{aligned}\right\} \Rightarrow \quad P(e|s_1) + P(e|s_M) = \Pr\left\{ |n| > d\sqrt{\tfrac{\mathcal{E}_g}{2}} \right\}$$

$$P(e|s_m) = \Pr\left\{ |n| > d\sqrt{\tfrac{\mathcal{E}_g}{2}} \right\}$$

$$P_M = \frac{1}{M}(M-2) \Pr\left\{ |n| > d\sqrt{\tfrac{\mathcal{E}_g}{2}} \right\} + \frac{1}{M} \Pr\left\{ |n| > d\sqrt{\tfrac{\mathcal{E}_g}{2}} \right\}$$

middle terms   two endpoints combined

$$P_M = \frac{M-1}{M} \Pr\left\{ |n| > d\sqrt{\tfrac{\mathcal{E}_g}{2}} \right\} = \frac{M-1}{M} \cdot 2 \int_{d\sqrt{\frac{\mathcal{E}_g}{2}}}^{+\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{n^2}{N_0}}\, dn$$

$$P_M = \frac{2(M-1)}{M} \cdot Q\left(\sqrt{\frac{d^2 \mathcal{E}_g}{N_0}} \right)$$