Optimum Receiver for Signals with Random Phase

- Oscillators are not phase synchronous
- Propagation delay is not known precisely

  - transmitted: $s(t) = Re\{g(t)\, e^{j\omega_c t}\}$
  - received: $s(t-t_0) = Re\{g(t-t_0)\, e^{j\omega_c (t-t_0)}\}$

$$= Re\{g(t-t_0)\, e^{-j\omega_c t_0}\, e^{j\omega_c t}\}$$

carrier shift due to propagation delay

$$\phi = -\omega_c t_0$$

large — large number — small

$f_c = 1\,MHz, \quad t_0 = 0.5\,\mu s \Rightarrow \phi = \pi$

- in some channels (radio channel) t_0 can change rapidly and randomly

- Idea: consider ϕ as a random variable.

Optimum Receiver for Binary Signals

$$s_m(t) = Re\left\{ \tilde{s}_m(t) e^{j\omega_c t} \right\} \qquad m=1,2 \qquad 0 \le t < T$$

$$\mathcal{E} = \int_0^T s_m^2(t)\,dt = \frac{1}{2}\int_0^T |\tilde{s}_m(t)|^2 dt$$

$$\rho_{12} \equiv \rho = \frac{1}{2\mathcal{E}} \int_0^T \tilde{s}_1^*(t)\,\tilde{s}_2(t)\,dt$$

$$n(t) = Re\left\{ (n_c(t) + j n_s(t)) e^{j\omega_c t} \right\} \qquad \text{— noise}$$

$$= Re\left\{ z(t)\, e^{j\omega_c t} \right\}$$

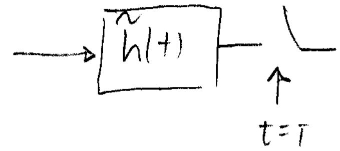
$$r(t) = Re\left\{ \left(\tilde{s}_m(t) e^{j\phi} + z(t) \right) e^{j\omega_c t} \right\}$$

$$\tilde{r}(t) = \tilde{s}_m(t) e^{j\phi} + z(t)$$

The optimum Demodulator

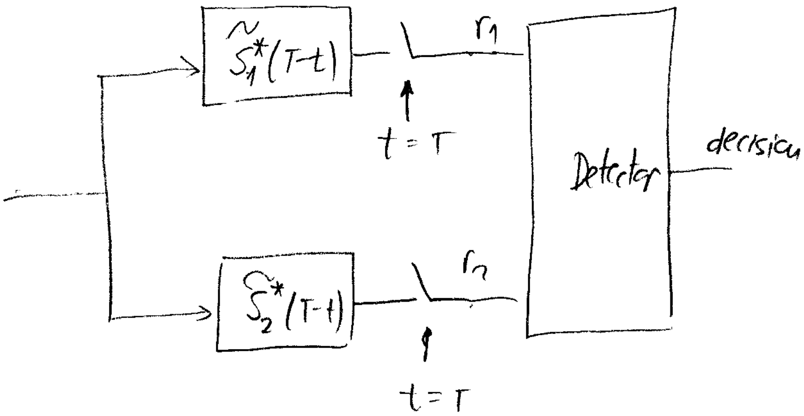
- Correlate with $\phi_n(t)$, orthonormal functions.
- Get sufficient statistics
- Work with complex envelopes:

$$\tilde{h}(t) = \tilde{s}^*(T-t)$$



$$\int_0^T |\tilde{s}(t)|^2 dt = 2\mathcal{E}$$

↑ output at $t=T$



$$r_m = r_{mc} + j r_{ms} \qquad m=1,2$$

Suppose that $s_1(t)$ was transmitted

$$r_1 = \int_0^T \tilde{r}(\tau) h_1(T-\tau)\,d\tau = \int_0^T \tilde{r}(\tau)\tilde{s}_1(T-(T-\tau))\,d\tau = \int_0^T \tilde{r}(t)\tilde{s}_1^*(\tau)\,d\tau$$

$$r_2 = \int_0^T \tilde{r}(\tau) h_2(T-\tau)\,d\tau = \int_0^T \tilde{r}(\tau)\tilde{s}_2(T-(T-\tau))\,d\tau = \int_0^T \tilde{r}(t)\tilde{s}_2^*(\tau)\,d\tau$$

$$r_1 = \int_0^T \tilde{r}(t)\tilde{s}^*(t)dt = \int_0^T \left(\tilde{s}_1(t)e^{j\phi} + z(t)\right)\tilde{s}_1^*(t)dt$$

$$= e^{j\phi}\int_0^T \tilde{s}_1(t)\tilde{s}_1^*(t)dt + \int_0^T z(t)\tilde{s}_1(t)dt$$

$$Re\left\{\int_0^T z(t)\tilde{s}_1(t)dt\right\} = n_{c1}$$

$$Im\left\{\int_0^T z(t)\tilde{s}_1(t)dt\right\} n_{s1}$$

$$= e^{j\phi}\cdot 2\mathcal{E} + n_{1c} + jn_{1s}$$

$$= 2\mathcal{E}\cos\phi + n_{1c} + j(2\mathcal{E}\sin\phi + n_{1s})$$

$$r_2 = \int_0^T \tilde{r}(t)\tilde{s}_2^*(t)dt = \int_0^T \left(\tilde{s}_1(t)e^{j\phi} + z(t)\right)s_2^*(t)dt$$

$$= e^{j\phi}\int_0^T \tilde{s}_1(t)\tilde{s}_2^*(t)dt + \int_0^T z(t)\tilde{s}_2(t)dt$$

$$Re\left\{\int_0^T z(t),\tilde{s}_2^*(t)dt\right\}$$

$$= e^{j\phi}\cdot 2\mathcal{E}\rho^* + n_{2c} + jn_{2s}$$

$$= e^{j\phi}\cdot 2\mathcal{E}|\rho|e^{-j\alpha_0} + n_{2c} + jn_{2s}$$

$$= 2\mathcal{E}|\rho|\cdot\cos(\phi-\alpha_0) + n_{2c} + j\left(2\mathcal{E}|\rho|\sin(\phi-\alpha_0) + n_{2s}\right)$$

The Optimum Detector

- Observes $r = ((r_{1c} + j r_{1s}), (r_{2c} + j r_{1s}))$
- Makes a decision by maximizing $P(\bar{s}_m | \bar{r})$ $m = 1, 2$

$$P(\bar{s}_m | \bar{r}) = \frac{p(\bar{r} | \bar{s}_m) P(\bar{s}_m)}{p(\bar{r})}$$

$$P(\bar{s}_1 | \bar{r}) \underset{S_2}{\overset{S_1}{\gtrless}} P(\bar{s}_2 | \bar{r})$$

$$N(\bar{r}) = \frac{p(\bar{r} | \bar{s}_1)}{p(\bar{r} | \bar{s}_2)} \underset{S_2}{\overset{S_1}{\gtrless}} \frac{P(\bar{s}_2)}{P(\bar{s}_1)}$$

likelihood ratio

ϕ is random

$$p(\bar{r} | \bar{s}_1) = \int_0^{2\pi} p(r | s_m, \phi)\, p(\phi)\, d\phi$$

Assume that signals are orthogonal $\rho = 0$

$$r_1 = r_{1c} + j r_{1s} = 2\mathcal{E} \cos\phi + n_{1c} + j(2\mathcal{E} \sin\phi + n_{1s})$$

$$r_2 = r_{2c} + j r_{2s} = n_{2c} + j n_{2s}$$

$n_{1c}, n_{1s}, n_{2c}, n_{2s}$ — mutually uncorrelated $\Rightarrow$ independent $N(0, \sigma^2)$

— why?

$$E(n_{1c} n_{1c}) = E\left(Re\left\{\int_0^T z(t)\tilde{s}_1(t)\,dt\right\} \cdot Re\left\{\int_0^T z(t)\tilde{s}_1(t)\,dt\right\}\right)$$

$Re\{z\} = \frac{1}{2}(z + z^*)$

$$= \frac{1}{4} E\left(\left(\int_0^T z(\mu) s^*(\mu)\,d\mu + \int_0^T z^*(t)\tilde{s}(t)\,dt\right)^2\right)$$

$$= \frac{1}{4} \cdot 2 \int_0^T \int_0^T E(z(\mu) z^*(t))\, \tilde{s}_1(t) \tilde{s}_1^*(\mu)\, d\mu\, dt$$

since $E(z(t) z(t+\tau)) = 0$
$E(z^*(t) z^*(t+\tau)) = 0$

$$= \frac{1}{4} \cdot 2 \cdot 2N_0 \underbrace{\int_0^T \tilde{s}_1(t) s_1^*(t)\,dt}_{2\mathcal{E}}$$

$$= 2N_0 \mathcal{E}$$

$$E(n_{1s} n_{1s}) = 2N_0 \mathcal{E}$$

$Re\{\int_0^T z(t)\tilde{s}_1(t)dt\} =$

$$E(n_{1c} n_{2c}) = \frac{1}{4} E\left(\left(\int_0^T z(t) s^*(t)\,dt + \int_0^T z^*(t)\tilde{s}_1(t)\,dt\right)\left(\int_0^T z(t) s_2^*(t) + \int_0^T z^*(t) \tilde{s}_2(t)\,dt\right)\right)$$

$$E(n_{1c} n_{1s}) = \frac{1}{4j} E\left(\left(\int_0^T z(t) s_1^*(t)dt + \int_0^T z^*(t)\tilde{s}_1(t)dt\right)\left(\int_0^T z(t) s_1^*(t)dt - \int_0^T z^*(t) s_1(t)dt\right)\right)$$

$Im\{z\} = \frac{1}{2j}(z - z^*)$

$$= \frac{1}{4j} \int_0^T \int_0^T E(z(t) z(\mu))\, s_1^*(t) s_1^*(t)\, dt\, d\mu - \frac{1}{4j} \int_0^T \int_0^T E(z^*(t) z^*(\mu))\, s_1(\mu) s_1^*(t)\, dt\, d\mu$$

$$= 0$$

Since r_{1c}, r_{1s}, r_{2c} and r_{2s} are independent

$$p(r_{1c}, r_{1s} | s_1, \phi) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left((r_{1c} - 2\mathcal{E}\cos\phi)^2 + (r_{1s} - 2\mathcal{E}\sin\phi)^2\right)}$$

$$p(r_{2c}, r_{2s}) = \frac{1}{2\pi\sigma^2} e^{-\frac{r_{2c}^2 + r_{2s}^2}{2\sigma^2}} \qquad \sigma^2 = 2N_0$$

The least favorable probability distribution for ϕ is $p(\phi) = \frac{1}{2\pi}$ $0 \le \phi \le 2\pi$

$$p(\bar{r} | \bar{s}_m) = \frac{1}{2\pi} \int_0^{2\pi} p(r_{1c}, r_{1s} | \bar{s}_1, \phi) d\phi$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{r_{1c}^2 + r_{1s}^2 + 4\mathcal{E}^2}{2\sigma^2}} \cdot \underbrace{\frac{1}{2\pi} \int_0^{2\pi} e^{\frac{1}{\sigma^2}(2\mathcal{E} r_{1c}\cos\phi + r_{1s}\sin\phi)} d\phi}_{I_0\left(\frac{2\mathcal{E}\sqrt{r_{1c}^2 + r_{1s}^2}}{\sigma^2}\right)}$$

$$p(r_{1c}, r_{1s} | \bar{s}_1) = \frac{1}{2\pi\sigma^2} e^{-\frac{\cancel{r_{1c}^2 + r_{1s}^2 + 4\mathcal{E}^2}}{2\sigma^2}} I_0\left(\frac{2\mathcal{E}\sqrt{r_{1c}^2 + r_{1s}^2}}{\sigma^2}\right)$$

$$p(r_{2c}, r_{2s} | \bar{s}_2) = \frac{1}{2\pi\sigma^2} e^{-\frac{r_{2c}^2 + r_{2s}^2 + 4\mathcal{E}^2}{2\sigma^2}} I_0\left(\frac{2\mathcal{E}\sqrt{r_{2c}^2 + r_{2s}^2}}{\sigma^2}\right)$$

$$\Lambda(\bar{r}) = \frac{p(r | \bar{s}_1)}{p(r | s_2)} = \frac{p(r_c\ r_s | \bar{s}_1)}{p(r_c\ r_s | \bar{s}_2)} = \ldots$$

$$p(r_{1c}, r_{1s}, r_{2c}, r_{2s} \mid \bar{s}_1) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(r_{1c}^2 + r_{1s}^2 + 4\mathcal{E}^2)} I_0\left(\frac{2\mathcal{E}\sqrt{r_{1c}^2 + r_{1s}^2}}{2\sigma^2}\right) \cdot \frac{1}{2\pi\sigma^2} e^{-\frac{r_{2c}^2 + r_{2s}^2}{2\sigma^2}}$$

$$p(r_{1c}, r_{1s}, r_{2c}, r_{2s} \mid \bar{s}_2) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(r_{1c}^2 + r_{1s}^2)} \cdot \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(r_{2c}^2 + r_{2s}^2 + 4\mathcal{E}^2)} I_0\left(\frac{2\mathcal{E}\sqrt{r_{2c}^2 + r_{2s}^2}}{2\sigma^2}\right)$$

$$\Lambda(\underline{r_{1c}\ r_{1s}\ r_{2c}\ r_{2s}}) = \frac{p(r_{1c}, r_{1s}, r_{2c}, r_{2s} \mid \bar{s}_1)}{p(r_{1c}, r_{1s}, r_{2c}, r_{2s} \mid \bar{s}_2)} = \frac{I_0\left(\frac{2\mathcal{E}\sqrt{r_{1c}^2 + r_{1s}^2}}{2\sigma^2}\right)}{I_0\left(\frac{2\mathcal{E}\sqrt{r_{2c}^2 + r_{2s}^2}}{2\sigma^2}\right)} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{P(s_2)}{P(s_1)}$$

If $P(s_1) = P(s_2)$:=

$I_0(x)$ is monotone

optimum detector: $\sqrt{r_{1c}^2 + r_{1s}^2} \underset{s_2}{\overset{s_1}{\gtrless}} \sqrt{r_{2c}^2 + r_{2s}^2}$ — Envelope detector!

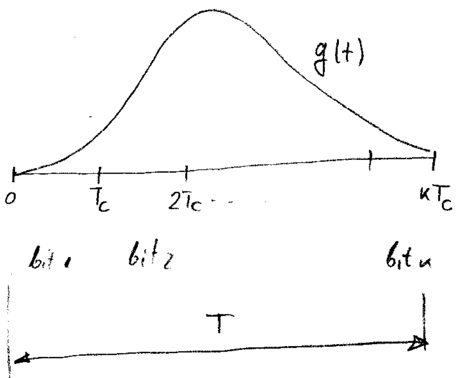
Comparison of Digital Modulation Methods

– Criterion :
- fix SNR
- fix the probability of error, P_e
- fix the bandwidth, W
- find the achievable rate R

$\Leftrightarrow$ find the ratio $\frac{R}{W}$ as a function of SNR for given P_e

$\frac{R}{W}$ – rate to bandwidth ratio (spectral efficiency)

PSK : M-ary modulation
- k-bits translated to a waveform
- bit rate R



$$T_c = \frac{1}{R}, \quad T = kT_c, \quad k = \log_2 M$$

$$W \cong \frac{1}{T} \Rightarrow W = \frac{1}{kT_c} = \frac{1}{\log_2 M \, T_c} = \frac{R}{\log_2 M}$$

$$\frac{R}{W} = \log_2 M$$

PAM : – a single sideband can be used

$$W \cong \frac{1}{2T} \Rightarrow \frac{R}{W} = 2\log_2 M$$

2x better than M-ary PSK

QAM :
- two orthogonal carriers, each carrier with one PAM signal
- The rate is doubled but double-sideband is required
- The same efficiency as PAM

$$\frac{R}{W} = 2\log_2 M$$

Orthogonal signals :

$M = 2^k$ orthogonal signals

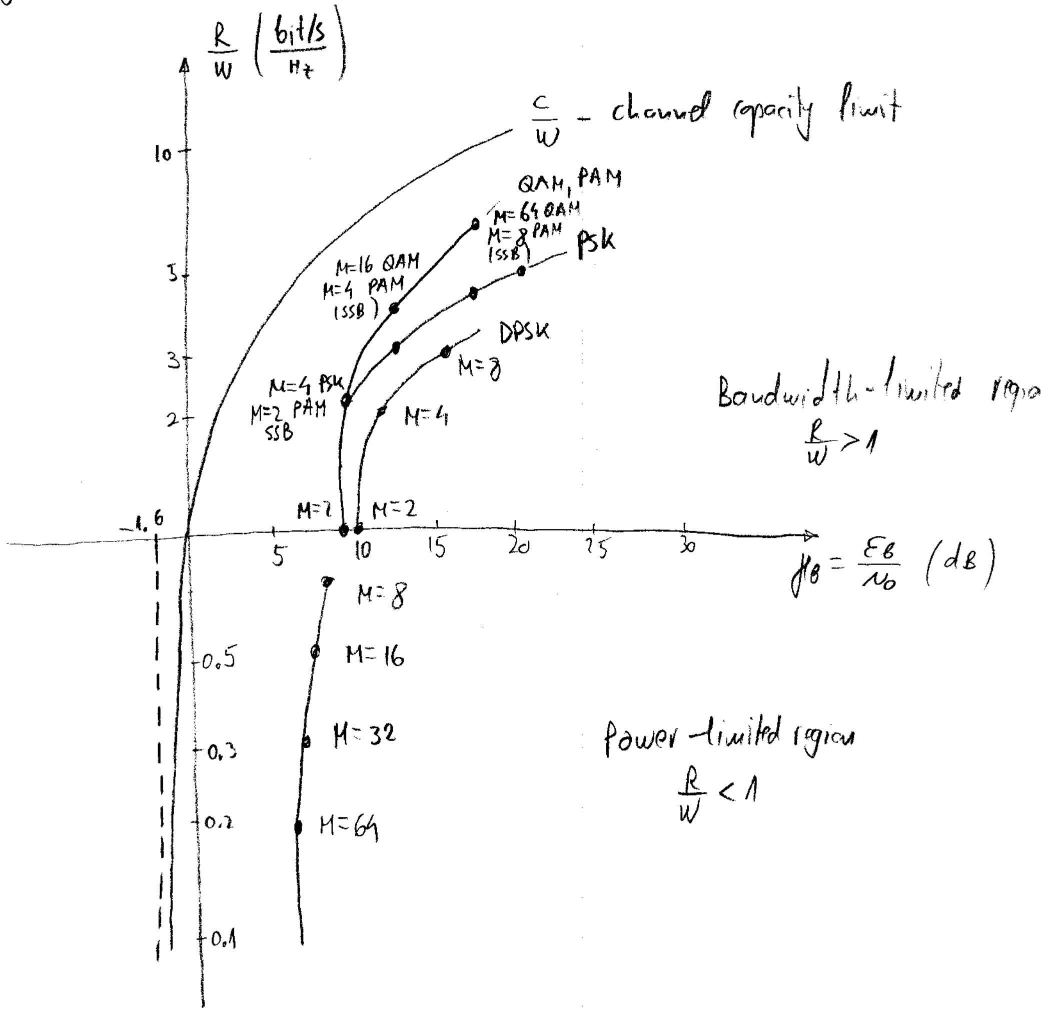
$$\rho_r = Re\{\rho_{km}\} = \frac{\sin 2\pi T(m-k)\Delta f}{2\pi T(m-k)\Delta f} = 0 \Rightarrow \Delta f = \frac{1}{2T}$$

- frequency separation between carriers - multiple of Δf
- the smallest separation $\frac{1}{2T}$
- The bandwidth is M times Δf

$$W = M \cdot \Delta f = M \frac{1}{2T} = \frac{M}{2\frac{k}{R}} = \frac{M}{2\log_2 M} \cdot R$$

$$\frac{R}{W} = \frac{2\log_2 M}{M}$$

Assume $P_M = 10^{-5}$



Conclusions:

PAM, QAM, PSK : – Increasing M results in higher $\frac{R}{W}$

– However the SNR must be increased.

M-ary orthogonal signals : – $\frac{R}{W} \leq 1$

– Increasing M results in lower $\frac{R}{W}$ due to an increase in the required bandwidth

– However SNR required to achive given P_M decreases as M increases.

– Appropriate for channels with large bandwidth

Coherent Detection of Differentially Encoded Schemes

- Receiver is not perfectly synchronized in phase and frequency to the incoming signal.
- The synchronization aquisition and tracking circuitry (carrier recovery circuitry) can lock onto one of finite number of phases.
- Information can be put into phase differences

1) Differentially encoded BPSK

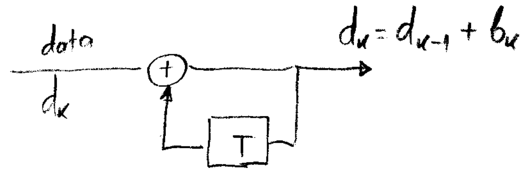
$$s_0(t) = \sqrt{\frac{2E_b}{T}} \cos \omega_c t$$

$$s_1(t) = -s_0(t)$$

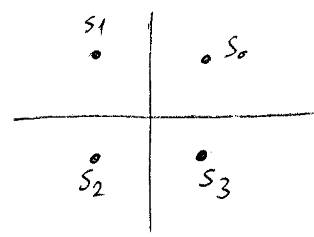
data :	1	0	1	0	0	0	1
phase shift :	π	0	π	0	0	0	π
transmitted phase :	π	π	0	0	0	0	π
transmitted signal :	s_1	s_1	s_0	s_0	s_0	s_0	s_0

$0 \rightarrow$ shift $= 0$

$1 \rightarrow$ shift $= \pi$



2) Differentially encoded QPSK



$$s_m(t) = \sqrt{\frac{2E_s}{T}} \cos\left(\omega_c t - \frac{\pi}{4} - m\frac{\pi}{2}\right)$$

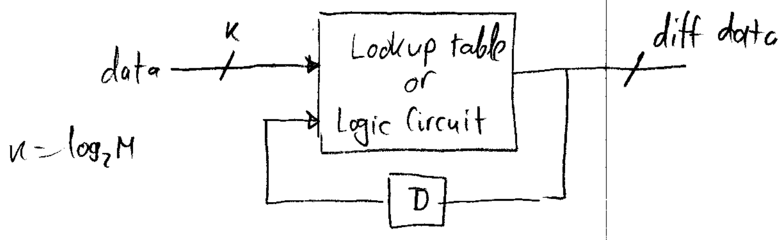
pairs of data bits are represented by phase differences

data :	01	00	11	11	10	01	00	01
phase shift :	$\pi/2$	0	$3\pi/2$	$3\pi/2$	π	$\pi/2$	0	$\pi/2$
transmitted phase :	$3\pi/4$	$3\pi/4$	$\pi/4$	$7\pi/4$	$3\pi/4$	$5\pi/4$	$5\pi/4$	$7\pi/4$
transmitted signal :	s_1	s_1	s_0	s_3	s_1	s_2	s_2	s_3

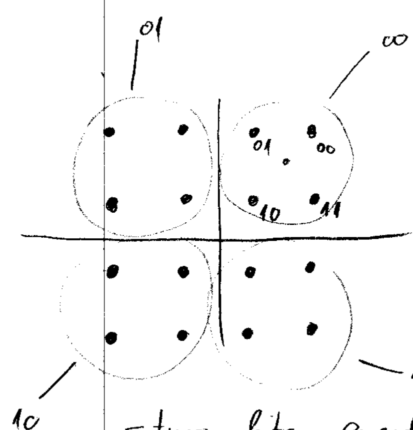
initial phase $\frac{\pi}{4}$

- QPSK may be treated as two independent BPSK channels and differential encoding may be implemented as two BPSK diff. encoder circuits operating in parallel

- In general Mary PSK diff encoder is



- Differentialy encoded 16-QAM

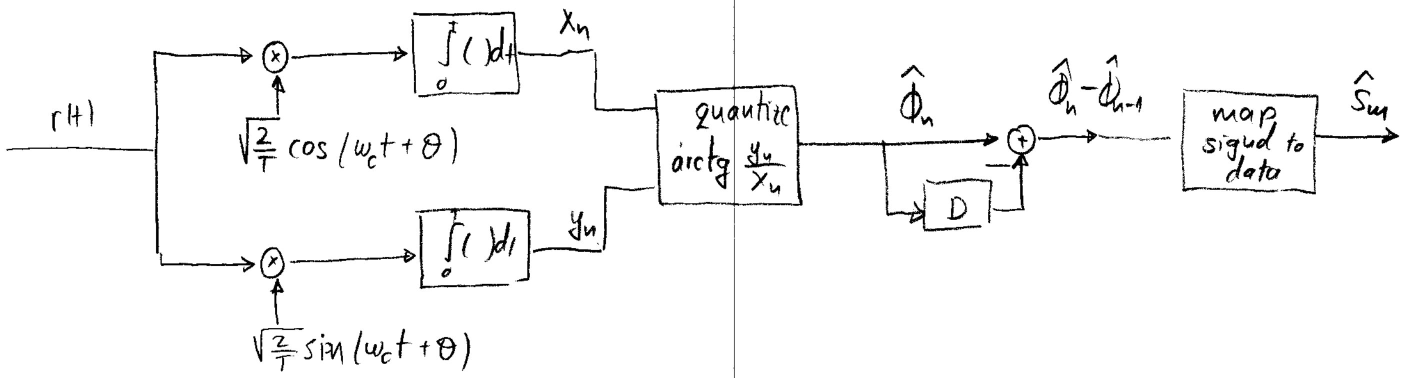


- two bits encoded differentialy (quadrants)
  two other bits indicate the point within a quadrant.

-

Receiver for Differentially Encoded M-PSK

$$S_m(t) = \sqrt{\frac{2E_s}{T}} \cos\left(\omega_c t + \left(\frac{\pi}{M} + \frac{2\pi m}{M}\right)\right) \qquad 0 \le m \le M-1$$



phase ambiguity

$$\theta \in \left\{ \frac{2\pi m}{M} : 0 \le m \le M-1 \right\}$$

$$\hat{\Phi}_n \in \left\{ \frac{\pi}{M} + 2\pi \frac{m}{M} : 0 \le m \le M-1 \right\}$$

Performance

$\hat{\phi}_n$ - n^{th}-detected phase
$\hat{b}_n$ - n^{th}-detected bit

$$\hat{b}_n = f(\hat{\phi}_n - \hat{\phi}_{n-1})$$

$$P_B = Pr\{\hat{b}_n \neq b_n\} = Pr\{\hat{\phi}_n \neq \phi_n\} \cdot Pr\{\hat{\phi}_{n-1} = \phi_{n-1}\}$$
$$+ Pr\{\hat{\phi}_n = \phi_n\} \cdot Pr\{\hat{\phi}_{n-1} \neq \phi_{n-1}\}$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\left(1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right) + \left(1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right) Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$= 2\, Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\left(1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right)$$

$$\cong 2\, Q\left(\sqrt{2\frac{E_b}{N_0}}\right)$$

$\Rightarrow$ differential encoding doubles the error rate relative to perfectly coherent reception. - a small price to pay.

- the alternative is to send a preamble to resolve phase ambiguities

- QPSK $\Leftrightarrow$ 2 BPSK

$$P_B(QPSK) = P_B(BPSK) = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

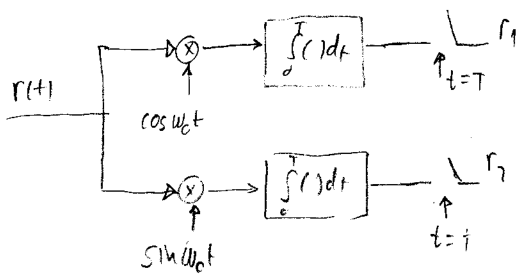
$$P_S(QPSK) = 1 - \underbrace{(1 - P_B(BPSK))^2}_{\text{probability that both bits in the symbol are correct}}$$

- M-ary PSK - more complicated

$$P_S(\text{M-ary PSK}) \cong 2\, P_S(\text{M-ary PSK - coherent})$$

Differential PSK (DPSK)

- Does not require the estimation of the carrier phase
- Received signal in any signaling interval is compared to the phase of the received signal from the preceding signaling interval.



$$\overline{r_k} = \left(\sqrt{\mathcal{E}_s}\cos(\theta_k - \phi) + n_{k1},\ \sqrt{\mathcal{E}_s}\sin(\theta_k - \phi) + n_{k2}\right)$$

$$\overline{r_k} = \sqrt{\mathcal{E}_s}\, e^{j(\theta_k - \phi)} + n_k$$ — noise vector $n_k = n_{1k} + j n_{2k}$

projections at k-th signalling interval

θ_k - phase angle of the transmitted signal (information)

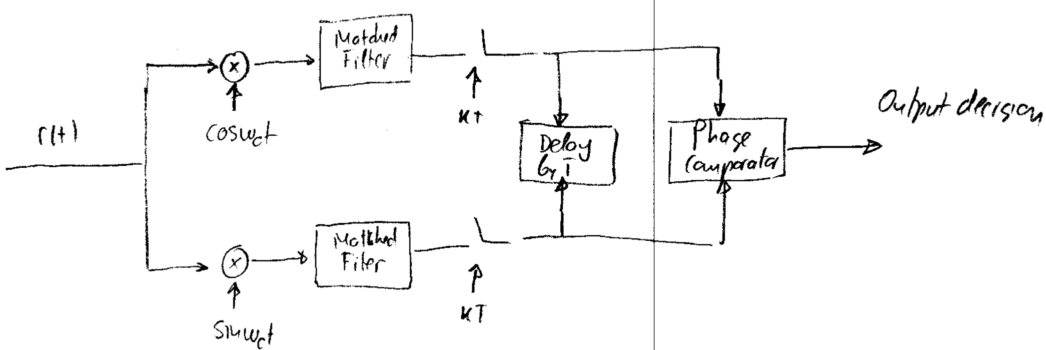
ϕ - carrier phase

Project r_k to r_{k-1}

$$r_k r_{k-1}^* = \mathcal{E}_s e^{j(\theta_k - \theta_{k-1})} + \sqrt{\mathcal{E}_s}\, e^{j(\theta_k - \phi)} n_{k-1}^* + \sqrt{\mathcal{E}_s}\, e^{-j(\theta_{k-1} - \phi)} n_k + n_k n_{k-1}^*$$

- in the absence of noise $n_k = n_{k-1} = 0$ $\Rightarrow$ $r_k r_{k-1}^* = \mathcal{E}_s e^{j(\theta_k - \theta_{k-1})}$

$\Rightarrow$ $E(r_k r_{k-1}^*)$ is not function of ϕ



Probability of error. – difficult except for M=2.
– dificult to find phase distribution of $r_k r_{k-1}^*$

Assume that $\theta_k - \theta_{k-1} = 0$

$$r_k r_{k-1}^* = \mathcal{E}_s e^{j(\theta_k - \theta_{k-1})} + \sqrt{\mathcal{E}_s}\, e^{j(\theta_k - \phi)} n_{k-1}^* + \sqrt{\mathcal{E}_s}\, e^{-j(\theta_{k-1} - \phi)} n_k + n_k n_{k-1}^*$$

$$= \mathcal{E}_s + \sqrt{\mathcal{E}_s}\, e^{j\theta_k} \underbrace{(n_{k-1}^* e^{-j\phi})}_{\text{new noise}} + \sqrt{\mathcal{E}_s}\, e^{-j\theta_{k-1}} \underbrace{(n_k e^{j\phi})}_{\text{new noise}} + n_k n_{k-1}'$$

statistics of the noise will not change if shifted by ϕ $\overset{\pm j\theta_k}{\Longrightarrow}$

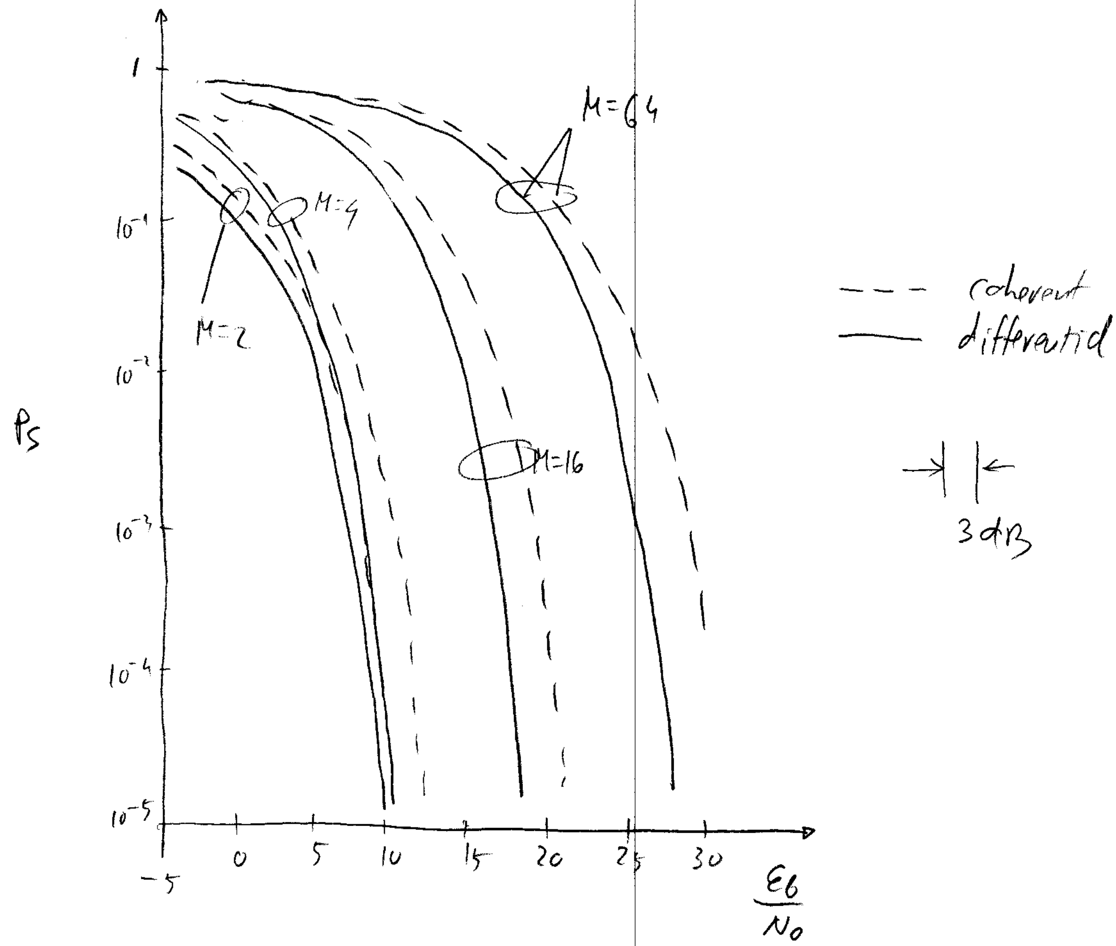
$$r_k r_{k-1}^* = \mathcal{E}_s + \sqrt{\mathcal{E}_s}\,(n_k + n_{k-1}^*) + \underbrace{n_k n_{k-1}^*}_{\text{small – neglect it.}}$$

$$\frac{r_k r_{k-1}^*}{\sqrt{\mathcal{E}_s}} = \sqrt{\mathcal{E}_s} + n_k + n_{k-1}^* = x + jy = R e^{j\theta_r}$$

$$x = \sqrt{\mathcal{E}_s} + \mathrm{Re}\{n_k + n_{k-1}^*\}$$

$$y = \mathrm{Im}\{n_k + n_{k-1}^*\}$$

$$\theta_r = \operatorname{arctg} \frac{y}{x}$$



1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
P_s
M=2
M=4
M=16
M=64
-5
0
5
10
15
20
25
30
$\frac{E_b}{N_0}$
coherent
differential
3 dB

Example of Noncoherent Detection – Binary FSK

- BFSK – orthogonal signals
- f_1, $f_1 + \Delta f = f_2$ – carrier frequencies $\Delta f = \frac{1}{2T}$ – minimum separation

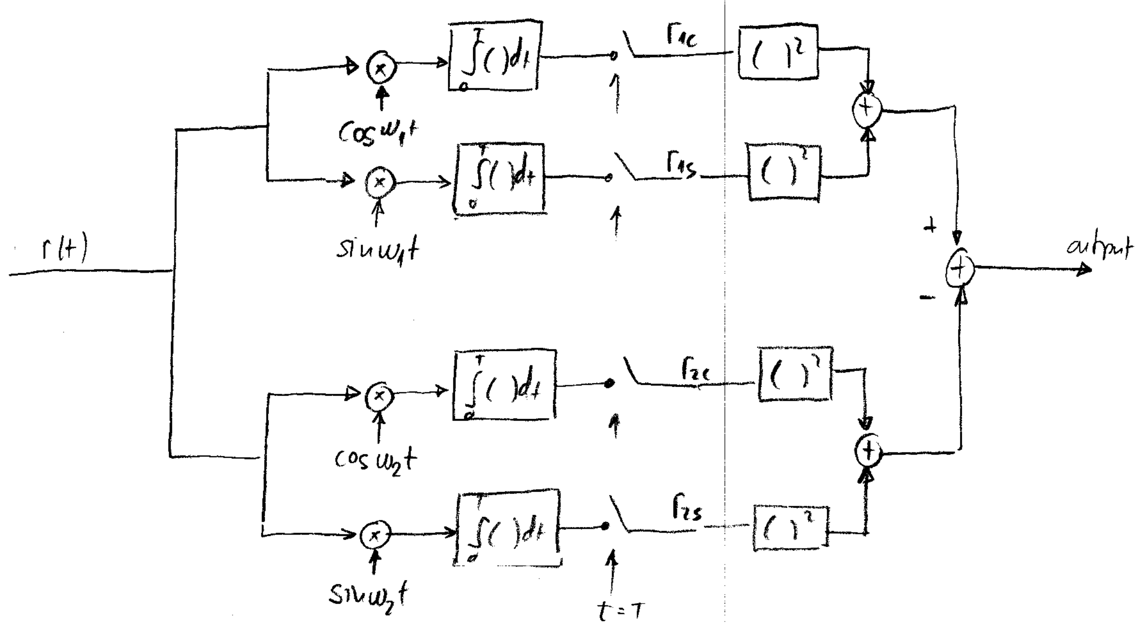
$$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos \omega_1 t \qquad \tilde{S}_1(t) = \sqrt{\frac{2E_b}{T_b}} \qquad 0 \le t < T_b$$

$$s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos \omega_2 t \qquad \tilde{S}_2(t) = \sqrt{\frac{2E_b}{T_b}} e^{j\Delta\omega t}$$

- Received signal $\quad r(t) = \sqrt{\frac{2E_b}{T_b}} \cos(\omega_m t + \phi_m) + n(t)$

ω_m, ϕ_m – angular frequency and phase of a carrier

- The receiver:



$$r_{kc} = \sqrt{\mathcal{E}_b}\,\frac{\sin (k-m)\Delta\omega T}{(k-m)\Delta\omega T}\cos\phi_m - \sqrt{\mathcal{E}_b}\,\frac{\cos(k-m)\Delta\omega T - 1}{(k-m)\Delta\omega T}\sin\phi_m + n_{kc}$$

$$r_{ks} = \sqrt{\mathcal{E}_b}\,\frac{\cos (k-m)\Delta\omega T - 1}{(k-m)\Delta\omega T}\cos\phi_m + \sqrt{\mathcal{E}_b}\,\frac{\sin(k-m)\Delta\omega T}{(k-m)\Delta\omega T}\sin\phi_m + n_{ks}$$

m-th signal was transmitted

$k = m$:

$$r_{mc} = \sqrt{\mathcal{E}_b}\cos\phi_m + n_{mc}$$
$$r_{ms} = \sqrt{\mathcal{E}_b}\sin\phi_m + n_{ms}$$

$k \neq m$: $\Delta f = \frac{1}{T}$

$$r_{kc} = n_{kc}$$
$$r_{ks} = n_{ks}$$

$\Rightarrow$ for envelope detection of FSK signals, the minimum frequency separation required for orthogonality of the signals is $\Delta f = \frac{1}{T}$

– twice as large as in phase-coherent case

Optimum Receiver for M-ary Orthogonal Signals

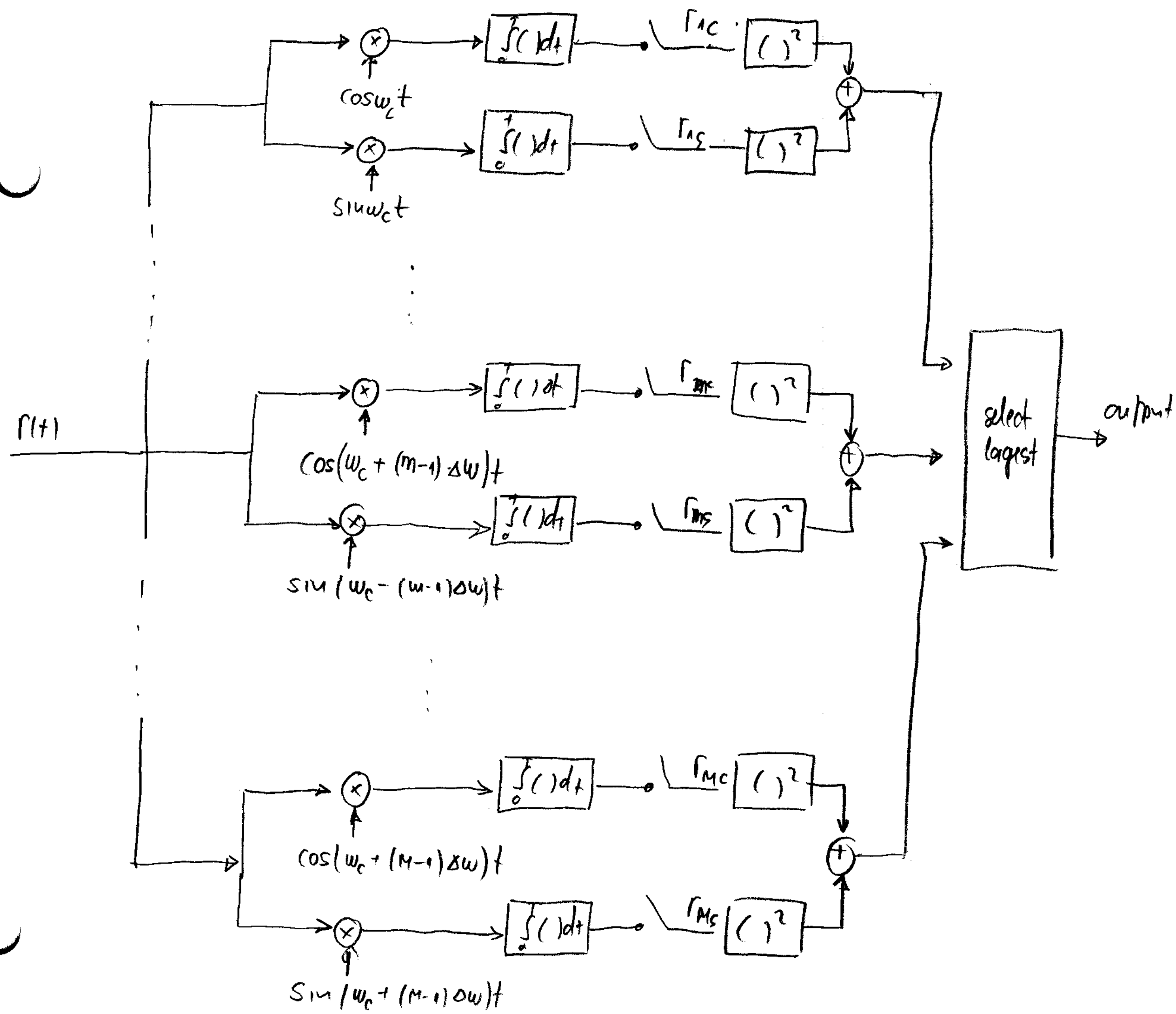
$$s_m(t) = Re\{\tilde{s}_m(t) e^{j w_c t}\} \qquad 1 \le m \le M \quad , \quad 0 \le t \le T$$

$\tilde{s}_m(t)$ – complex envelopes, equal energies

$$r_m = r_{m,c} + j r_{m,s} = \int_0^T \tilde{r}(t) \tilde{s}_m^*(t)\, dt \qquad 1 \le m \le M$$

$$|r_m| = \sqrt{r_{mc}^2 + r_{ms}^2}$$

$$\Delta f = \frac{1}{T}$$



Probability of Error for Envelope Detection of M-ary Orthogonal Signals