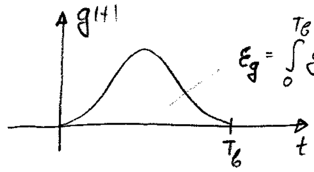
Performance of The Optimum Receiver for Memoryless Modulation

PAM: Binary

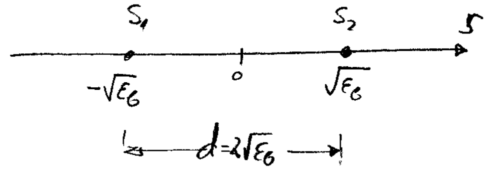
$s_1(t) = g(t)$

$s_2(t) = -g(t)$

antipodal signalling



$\mathcal{E}_g = \int_0^{T_b} g^2(t)\,dt$

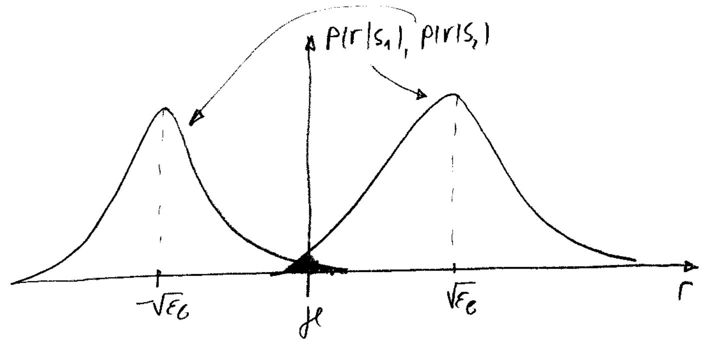


Let us assume that $P(s_1) = P(s_2)$
and that $s_1(t)$ was transmitted

$$r = s_1 + n = \sqrt{\mathcal{E}_b} + n \qquad \sigma_n^2 = \frac{1}{2} N_0$$

$$p(r|s_1) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-\sqrt{\mathcal{E}_b})^2}{N_0}}$$

$$p(r|s_2) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-\sqrt{\mathcal{E}_b})^2}{N_0}}$$



$$P_e = P(s_1)\cdot P(s_2|s_1) + P(s_2)\,P(s_1|s_2)$$

$$P_e = \frac{1}{2}\cdot 2\cdot P(s_2|s_1) = P(r<\mu) = \int_{-\infty}^{\mu} p(r|s_1)\,dr$$

optimal μ: $\mu = 0$

$$P_e = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{0} e^{-\frac{(r-\sqrt{\mathcal{E}_b})^2}{N_0}}\,dr$$

$$P_e = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{-\sqrt{\frac{2\mathcal{E}_b}{N_0}}} e^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2\mathcal{E}_b}{N_0}}}^{+\infty} e^{-x^2/2}\,dx$$

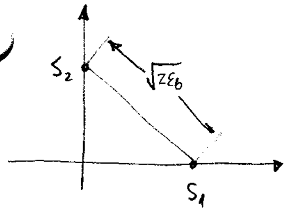
$$P_b = P_e = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

$\frac{\mathcal{E}_b}{N_0}$ – signal to noise ratio per bit

$d = 2\sqrt{\mathcal{E}_b}$ – distance

$$P_b = Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$$

Orthogonal signals



$S_1 = (\sqrt{\mathcal{E}_b}, 0)$
$S_2 = (0, \sqrt{\mathcal{E}_b})$
$d = \sqrt{2\mathcal{E}_b}$

Assume that s_1 was transmitted.

$$\bar{r} = (\sqrt{\mathcal{E}_b} + n_1, n_2)$$

$$c(\bar{r}, \bar{s}_1) = 2\bar{r}\bar{s}_1 - \|s_1\|^2 = 2(\sqrt{\mathcal{E}_b} + n_1, n_2)\cdot(\sqrt{\mathcal{E}_b}, 0) - \mathcal{E}_b = 2\mathcal{E}_b + 2\sqrt{\mathcal{E}_b}\, n_1 - \mathcal{E}_b = \mathcal{E}_b + 2\sqrt{\mathcal{E}_b}\, n_1$$
$$c(\bar{r}, \bar{s}_2) = 2\bar{r}s_2 - \|s_2\|^2 = 2(\sqrt{\mathcal{E}_b} + n_1, n_2)(0, \sqrt{\mathcal{E}_b}) - \mathcal{E}_b = 2\sqrt{\mathcal{E}_b}\, n_2 - \mathcal{E}_b = -\mathcal{E}_b + 2\sqrt{\mathcal{E}_b}\, n_2$$

$$P(e|\bar{s}_1) = P_r\{c(\bar{r},\bar{s}_2) > c(\bar{r},\bar{s}_1)\}$$
$$= P_r\{n_2 - \tfrac{1}{2}\sqrt{\mathcal{E}_b} > n_1 + \tfrac{1}{2}\sqrt{\mathcal{E}_b}\}$$
$$= P_r\{n_2 - n_1 > \sqrt{\mathcal{E}_b}\}$$
$$= \frac{1}{\sqrt{2\pi N_0}} \int_{\sqrt{\mathcal{E}_b}}^{+\infty} e^{-\frac{x^2}{2N_0}}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{\mathcal{E}_b}{N_0}}}^{+\infty} e^{-\frac{x^2}{2}}\, dx$$
$$= Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$$
$$P_b = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$$

$n_1 : N(0, \frac{N_0}{2})$
$n_2 : N(0, \frac{N_0}{2})$
$n_2 - n_1 : N(0, N_0)$

antipodal	orthogonal
$P_b = Q\left(\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right)$	$P_b = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$
$d = 2\cdot\sqrt{\mathcal{E}_b}$	$d = \sqrt{2\mathcal{E}_b}$

orthogonal signals require 2x increase of energy to achieve the same P_b

$10\log_{10} 2 = 3\,dB$ – SNR loss is 3dB.

why? – the distance is smaller

Probability of Error for M-ary Orthogonal Signals

- equal-energies, equal probabilities
- s_m $1 \le m \le M$

$$C(\bar{r}, \bar{s}_m) = 2\bar{r}\bar{s}_m - \|\bar{s}_m\|^2$$
$$C'(\bar{r}, \bar{s}_m) = 2\bar{r}\bar{s}_m$$

- Assume the s_1 was transmitted

$$r = (\sqrt{\mathcal{E}_s} + n_1, n_2, \dots, n_M) \qquad \sigma_n = \tfrac{1}{2}N_0, \quad n_1, n_2, \dots, n_M - \text{independent}, \ N(0, \tfrac{N_0}{2})$$

$$C'(\bar{r}, \bar{s}_1) = (\sqrt{\mathcal{E}_s} + n_1, n_2, \dots, n_M)(\sqrt{\mathcal{E}_s}, 0, 0, \dots, 0) = \mathcal{E}_s + \sqrt{\mathcal{E}_s}\, n_1$$
$$C'(\bar{r}, \bar{s}_2) = (\sqrt{\mathcal{E}_s} + n_1, n_2, \dots, n_M)(0, \sqrt{\mathcal{E}_s}, \dots, 0) = \sqrt{\mathcal{E}_s}\, n_2$$
$$\vdots$$
$$C'(\bar{r}, \bar{s}_M) = (\sqrt{\mathcal{E}_s} + n_1, n_2, \dots, n_M)(0, 0, \dots, \sqrt{\mathcal{E}_s}) = \sqrt{\mathcal{E}_s}\, n_M$$

Correlator outputs:

$$y_1 = \sqrt{\mathcal{E}_s} + n_1 \quad \Rightarrow \quad p_{Y_1}(y_1) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y_1 - \sqrt{\mathcal{E}_s})^2}{N_0}}$$
$$y_2 = n_2$$
$$y_M = n_M \qquad p_{Y_m}(y_m) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{y_m^2}{N_0}} \qquad 2 \le m \le M$$

Probability of correct decision

$$P_c = \int_{-\infty}^{+\infty} p(y_1)\, P(n_2 < y_1, n_3 < y_1, \dots, n_M < y_1 \mid y_1)\, dy_1$$

all other samples in other dimensions should be smaller than y_1

$$P(n_m < y_1 \mid y_1) = \int_{-\infty}^{y_1} p_{Y_m}(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y_1\sqrt{2/N_0}} e^{-\frac{x^2}{2}} dx$$

$$P_c = \int_{-\infty}^{+\infty} \prod_{m=2}^{M} P(n_m < y_1 \mid y_1)\, p(y_1)\, dy_1 = \int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y_1\sqrt{2/N_0}} e^{-\frac{x^2}{2}} \right)^{M-1} p(y_1)\, dy_1$$

$P_M = 1 - P_c$ - probability of error

$$P_M = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(1 - \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}} dx \right)^{M-1} \right) e^{-\frac{1}{2}\left(y - \sqrt{\frac{2E_s}{N_0}}\right)^2} dy$$

symbol error probability

P_M

$E_s = K \cdot E_B \qquad 2^K = M$

energy per symbol

energy per bit

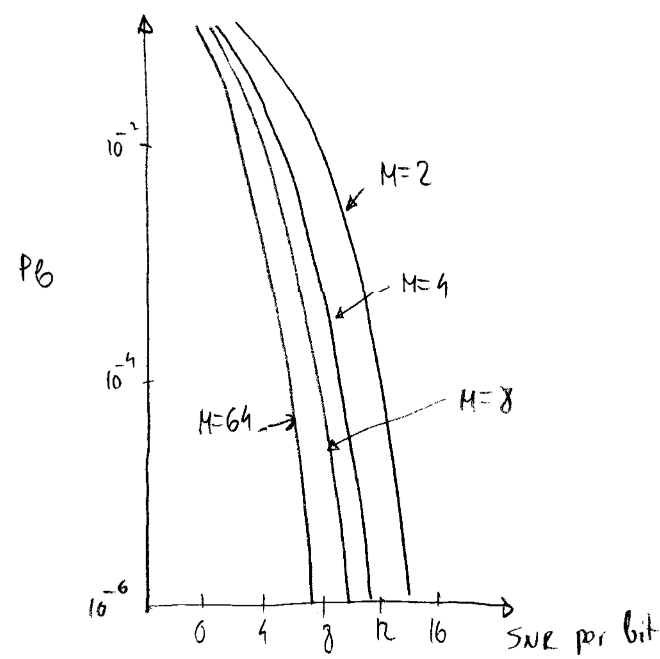
$$\text{particular symbol error probability} = \frac{P_M}{M-1}$$

$\binom{K}{j}$ – combinations in which j bits out of K may be in error

The average number of bit errors per symbol (k-bits)

$$\sum_{j=0}^{K} j \binom{K}{j} \frac{P_M}{M-1} = K \frac{2^{K-1}}{2^K - 1} P_M$$

$$P_B = \frac{2^{K-1}}{2^K - 1} P_M \cong \frac{P_M}{2} \qquad k \gg 1$$



Union Bound on the Probability of Error

What happens when M increases

E_i - event that $C(\bar{r}, \bar{s}_i) > C(\bar{r}, \bar{s}_1)$ $\quad i \neq 1$

$$P_M = P\left(\bigcup_{i=1}^{M-1} E_i\right) \le \sum_{i=1}^{M-1} P(E_i)$$

$$P_M \le (M-1) P_2 = (M-1)\, Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}}\right) < M Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}}\right)$$

$$Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}}\right) < e^{-\frac{\mathcal{E}_s}{2N_0}}$$

$$P_M < M e^{-\frac{\mathcal{E}_s}{2N_0}} = 2^k e^{-\frac{k\mathcal{E}_b}{2N_0}}$$

$$< e^{-k(\mathcal{E}_b/N_0 - 2\ln 2)}$$

$k \to \infty$ $(M \to \infty)$ $\quad P_M \to 0$ if $\frac{\mathcal{E}_b}{N_0} > 2\ln 2$

$$\frac{\mathcal{E}_b}{N_0} > 2\ln 2 = 1.42\ dB$$

Not very tight bound

$$P_M < 2 e^{-k\left(\sqrt{\frac{\mathcal{E}_b}{N_0}} - \sqrt{\ln 2}\right)^2}$$

$$\frac{\mathcal{E}_b}{N_0} > \ln 2 = 0.693 = -1.6\ dB$$

Shannon limit