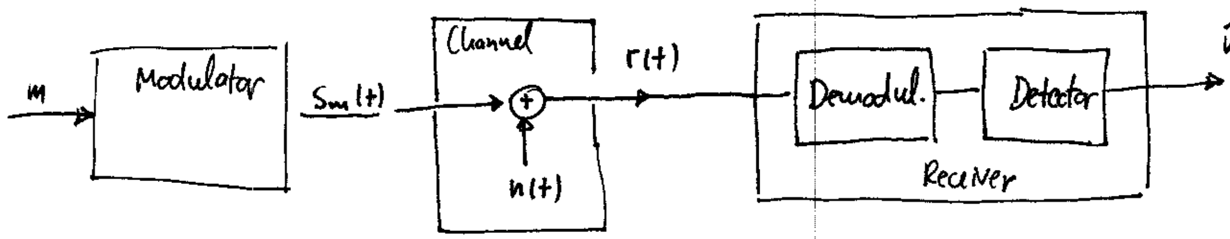
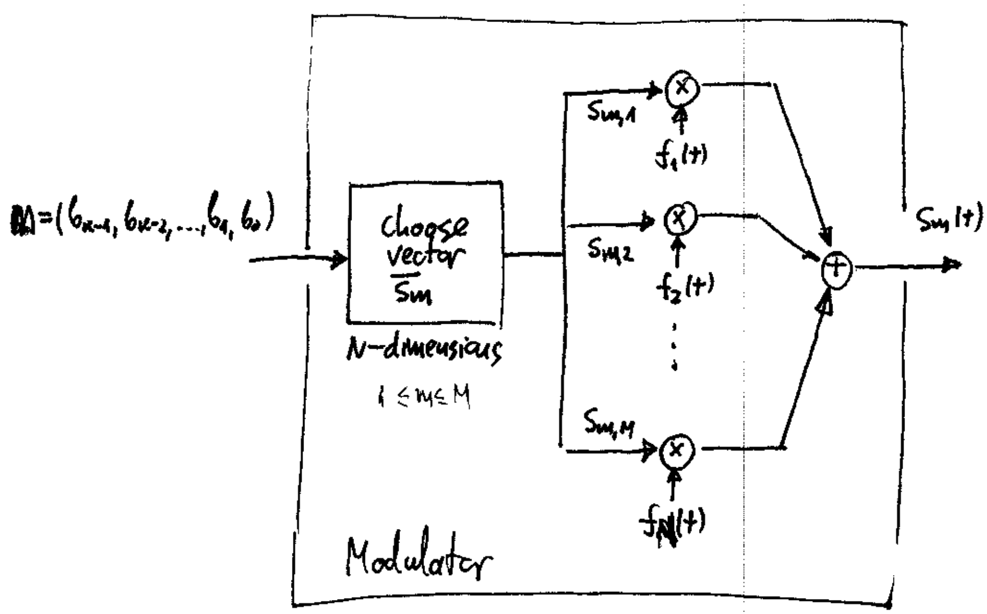
Optimum Receivers for the AWGN Channel

- The transmitter sends digital information by use of M signal waveforms: $\{ s_m(t) : 1 \le m \le M \}$.
- Each waveform is transmitted within the signaling (symbol) interval T
- We consider transmission of information in $0 \le t \le T$





- $n(t)$: AWGN process with power spectral density $\Phi_{nn}(f) = \frac{1}{2} N_0 (W/Hz)$
- Based on the observation $r(t)$ over $0 \le t \le T$ we wish to design a receiver that is optimal in the sense that it minimizes probability of error.
- Demodulator – converts $r(t)$ to $\bar{r} = (r_1, \ldots r_N)$
- Detector: – decides which one of M possible waveforms was sent.

Correlation Demodulator – Projection Demodulator

– The signal and noise are expanded into a series of orthonormal basis functions, i.e.

$$(r(t), \phi_k(t)) = \int_0^T r(t)\,\phi_k(t)\,dt = \int_0^T (s_m(t)+n(t))\,\phi_k(t)\,dt = \int_0^T s_m(t)\,\phi_k(t)\,dt + \int_0^T n(t)\,\phi_k(t)\,dt$$

$r_k = (r(t), \phi_k(t))$

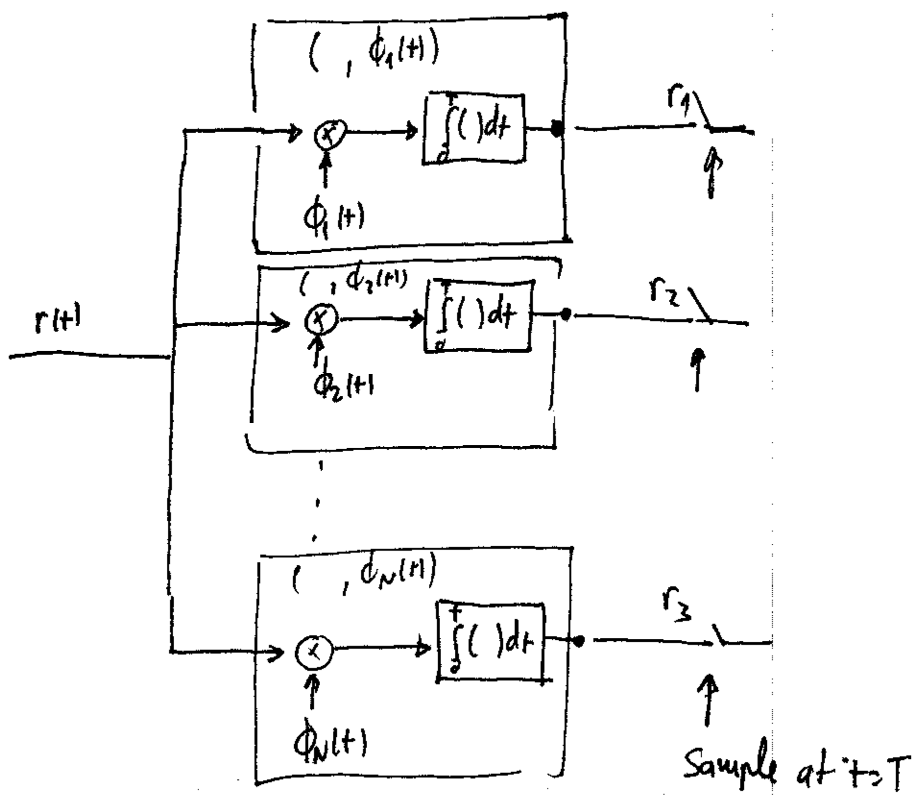
$s_{m,k} = (s_m(t), \phi_k(t))$ ↙ projections

$n_k = (n(t), \phi_k(t))$ $\longrightarrow$ $r_k = s_{m,k} + n_k \qquad 1 \le k \le N$

$$r(t) = \sum_{k=1}^{N} r_k \cdot \phi_k(t) + \varepsilon(t)$$ error of approximation

The dimension of $n(t)$ is larger than N

$$\varepsilon(t) = n(t) - \sum_{k=1}^{N} n_k \phi_k(t)$$



Demodulator

Demodulation:

$$r(t) \neq \sum_{k=1}^{N} r_k \phi_k(t) \quad - \text{ Can } r_k \text{ be used to make the decision ?}$$

$$n(t) = \sum_{k=1}^{+\infty} n_k \phi_k(t) = \sum_{k=1}^{N} n_k \phi_k(t) + \varepsilon(t)$$

$$E(\varepsilon(t)\cdot r_k) = E(\varepsilon(t)\cdot s_{m,k} + \varepsilon(t)\cdot n_k)$$

$$= \underbrace{E(\varepsilon(t))}_{=0}\cdot \underset{\text{constant}}{s_{m,k}} + E(\varepsilon(t)\cdot n_k)$$

$$= E(\varepsilon(t)\cdot n_k)$$

$$= E\left(\left(n(t) - \sum_{j=1}^{N} n_j \phi_j(t)\right) n_k\right)$$

$$= E\left(n(t)\cdot \int_0^T n(\tau)\phi_k(\tau)d\tau - \sum_{j=1}^{N} n_j n_k \phi_j(t)\right)$$

$$= \int_0^T \underbrace{E(n(t)n(\tau))}_{\frac{N_0}{2}\delta(t-\tau)}\phi_k(\tau)d\tau - \sum_{j=1}^{N} \underbrace{E(n_j n_k)}_{\delta_{j,k}\frac{N_0}{2}} \phi_j(t)$$

$$= \frac{1}{2} N_0 \phi_k(t) - \frac{1}{2} N_0 \phi_k(t)$$

$$= 0$$

$\Rightarrow$ $\varepsilon(t)$ and r_k are uncorrelated (1)

$$E(n_k n_j) = \int_0^T \int_0^T E(n(t)n(\tau))\, \phi_k(t)\, \phi_j(t)\, dt\, d\tau$$

$$= \frac{1}{2} N_0 \int_0^T \int_0^T \delta(t-\tau)\, \phi_k(t)\, \phi_j(\tau)\, dt\, d\tau$$

$$= \frac{1}{2} N_0 \int_0^T \phi_k(t)\, \phi_j(t)\, dt$$

$$= \frac{1}{2} N_0\, \delta_{m,k}$$

$\Rightarrow$ n_k are uncorrelated $\Rightarrow$ n_k are independent

· n_k are zero mean, Gaussian

$$E(r_k) = E(s_{m,k} + n_k) = s_{m,k}$$

$$\sigma_r^2 = \sigma_n^2 = \frac{1}{2} N_0$$

— r_k are Gaussian (2)

$$p(\bar{r} \mid \bar{s}_m) = \prod_{k=1}^{N} p(r_k \mid s_{m,k}) \qquad 1 \le m \le M$$

$$p(r_k \mid s_{m,k}) = \frac{1}{\sqrt{2\pi\sigma_n^2}}\, e^{-\frac{(r_k - s_{m,k})^2}{2\sigma_n^2}} \qquad 1 \le k \le N$$

$$p(\bar{r} \mid \bar{s}_m) = \frac{1}{(\pi N_0)^{\frac{N}{2}}} \cdot e^{-\frac{1}{N_0}\sum_{k=1}^{N}(r_k - s_{m,k})^2}$$

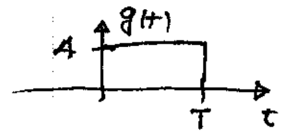
r_k and $\varepsilon(t)$ — are ~~uncorr~~ uncorrelated and Gaussian $\Rightarrow$ r_k and $\varepsilon(t)$ are independent

$\Rightarrow$ $\varepsilon(t)$ does not help to learn more about $r(t)$

$\Rightarrow$ $\varepsilon(t)$ is irrelevant to the decision making.

Example:

M-ary PAM



$g(t)$ - rectangular $\quad \mathcal{E}_g = \int_0^T g^2(t) = A^2 T$

energy

$N = 1. \quad \phi_1(t) = \frac{1}{\sqrt{\mathcal{E}_g}} g(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$

$$r_1 = (r(t), \phi_1(t)) = \frac{1}{\sqrt{T}} \int_0^T r(t)\, dt$$

square

$$r_1 = \frac{1}{\sqrt{T}} \int_0^T (s_m(t) + n(t))\, dt \cdot 1$$

$$r_1 = \frac{1}{\sqrt{T}} \int_0^T s_m(t)\, dt + \frac{1}{\sqrt{T}} \int_0^T n(t)\, dt$$

$$r_1 = s_m + n$$

$$\sigma_n^2 = E\left(\frac{1}{T} \int_0^T \int_0^T n(t)\, n(\tau)\, dt\, d\tau \right) = \frac{N_0}{2T} \int_0^T \int_0^T \delta(t-\tau)\, dt\, d\tau = \frac{1}{2} N_0$$

$$p(r_1 | s_m) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r - s_m)^2}{N_0}}$$

Matched-Filter Demodulator

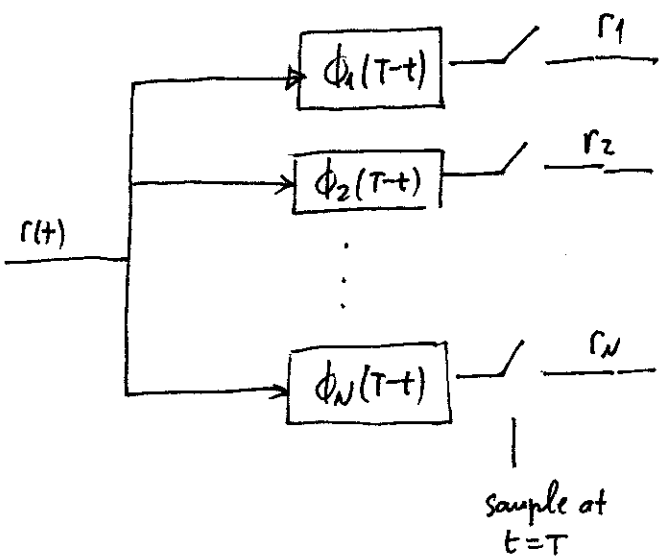
$$r_k = \int_0^T r(t)\,\phi_k(t)\,dt$$

$$r_k(t) = \int_0^t r(\tau)\,\phi_k(T+\tau-t)\,d\tau, \quad r_k = r_k(T)$$

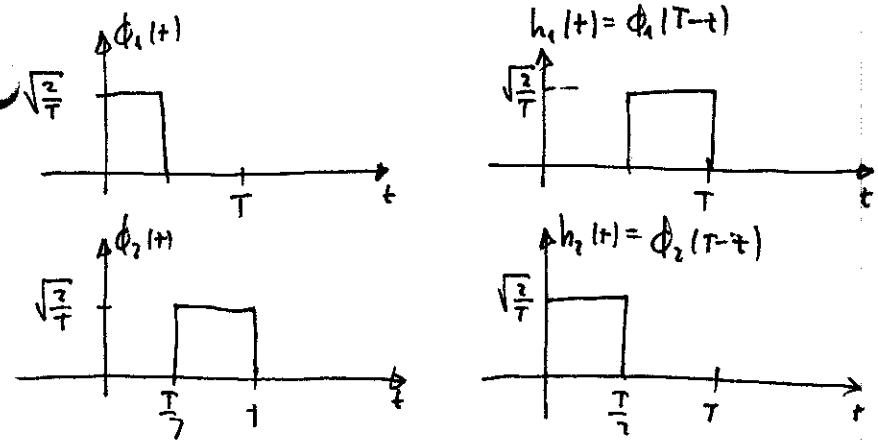
$$r_k(t) = \int_0^t r(\tau)\,\underbrace{\phi_k(T+\tau-t)}_{h_k(t-\tau)}\,d\tau \quad \Rightarrow \quad r_k(t) = r(t) * \phi_k(T-t)$$

$$h_k(t-\tau) = \phi_k(T+\tau-t)$$

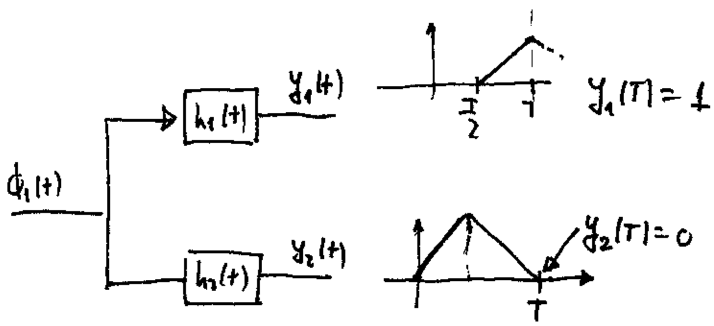
$$h_k(t) = \phi_k(T-t)$$

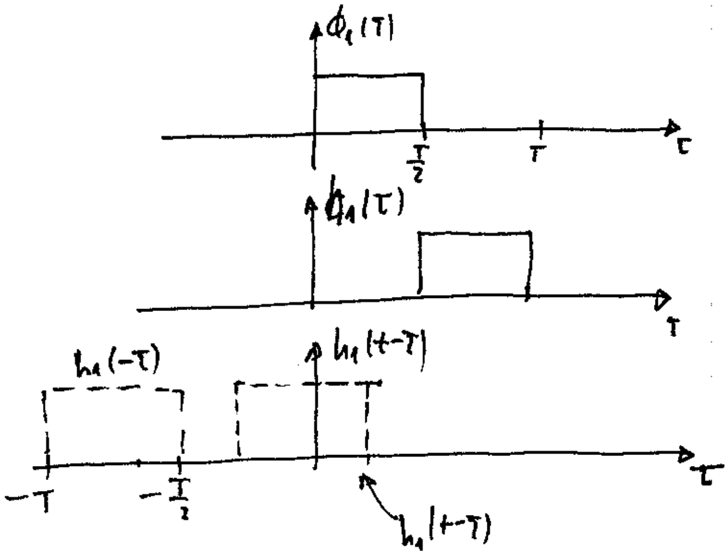


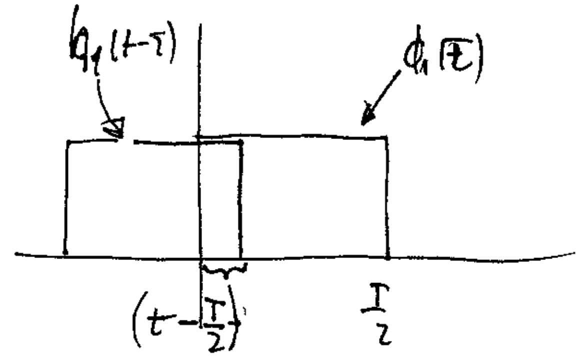
Example



— no noise







$t < T/2 \qquad \phi_1(t) \cdot h_1(t-\tau) = 0$