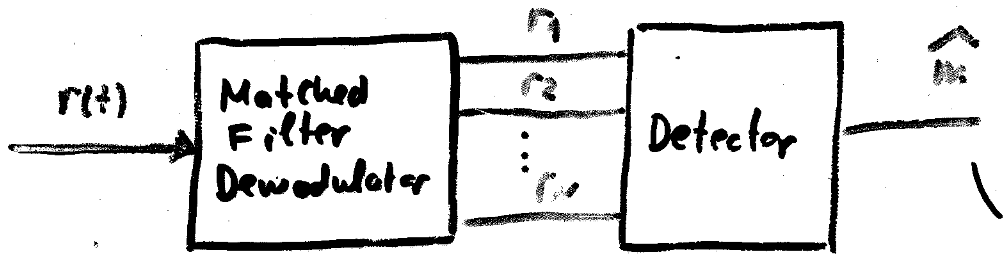
The Optimum Detector

- In AWGN channel correlation or matched filter demodulator produces the vector $r = (r_1, \dots, r_N)$ – sufficient statistics



which one of M possible signals $s_m(t)$ has been transmitted

- What is optimal Detector?
- assume memoryless channel

- a posteriory probabilities: $p(\bar{s}_m \text{ was transmitted} \mid \bar{r}) = P(\bar{s}_m \mid \bar{r})$

MAP criterion: select $\bar{s}_m$ that maximizes $P(\bar{s}_m \mid \bar{r})$

Maximum Aposterrory Criterion

Using Bayes' rule

$$P(\bar{s}_m \mid \bar{r}) = \frac{P(\bar{s}_m) \cdot p(\bar{r} \mid \bar{s}_m)}{p(\bar{r})}$$

$$P(\bar{s}_m \mid \bar{r}) = \frac{P(\bar{s}_m)\, p(\bar{r} \mid \bar{s}_m)}{\sum_{m=1}^{M} P(\bar{s}_m)\, p(\bar{r} \mid \bar{s}_m)}$$

conditional PDF (Channel)

a priory probabilities

Computation of $P(\bar{s}_m|\bar{r})$ requires knowledge of: $P(\bar{s}_m)$

: $p(\bar{r}|\bar{s}_m)$

Special cases: $P(\bar{s}_m) = \frac{1}{M}$ - all signals are equally probable

$$P(\bar{s}_m|\bar{r}) = \frac{1}{M\,p(\bar{r})} \cdot \underbrace{p(\bar{r}|\bar{s}_m)}_{\text{relevant part}}$$

independent of $\bar{s}_m$

$p(\bar{r}|\bar{s}_m)$ — likelihood function

ML criterion: maximum likelihood criterion

$$ML = MAP \quad \text{if} \quad P(s_m) = \frac{1}{M}$$

AWGN :

$$p(\bar{r}|\bar{s}_m) = \frac{1}{(\pi N_0)^{\frac{N}{2}}} e^{-\frac{1}{N_0}\sum_{k=1}^{N}(r_k - s_{m,k})^2}$$

log likelihood (log is a monotonic function)

$$\ln p(\bar{r}|\bar{s}_m) = -\frac{N}{2}\ln(\pi N_0) - \frac{1}{N_0}\underbrace{\sum_{k=1}^{N}(r_k - s_{m,k})^2}_{D(\bar{r},\bar{s}_m) \leftarrow \text{Euclidean distance}}$$

$D(\bar{r},\bar{s}_m)$, $1 \le m \le M$ - distance metrics

ML rule $\Leftrightarrow$ minimum distance detection

$$D(\bar{r},\bar{s}_m) = \sum_{k=1}^{N} r_k^2 - 2\sum_{k=1}^{N} r_k s_{m,k} + \sum_{k=1}^{N} s_{m,k}^2$$

$$= \underbrace{\|r\|^2}_{\text{Equal for all } m} - 2\,\bar{r}\cdot\bar{s}_m + \|s_m\|^2 \qquad 1 \le m \le M$$

$$D'(\bar{r},\bar{s}_m) = -2\,\bar{r}\cdot\bar{s}_m + \|s_m\|^2$$ - minimize $D'(\bar{r},\bar{s}_m)$

PAM :

- Binary PAM
- $S_1 = \sqrt{E_b}$
- $S_2 = -\sqrt{E_b}$ $\quad$ E_b - energy per bit

$P(S_1) = p$
$P(S_2) = 1 - p$

$r = \pm\sqrt{E_b} + n \qquad n : N(0, \frac{1}{2}N_0)$

$$p(r|s_1) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(r-\sqrt{E_b})^2}{2\sigma^2}} \qquad p(r|s_2) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(r+\sqrt{E_b})^2}{2\sigma^2}}$$

$$\frac{p(r|s_1)\,P(s_1)}{p(r|s_2)\,P(s_2)} \underset{S_2}{\overset{S_1}{\gtrless}} 1$$

$$\frac{p(r|s_1)}{p(r|s_2)} \underset{S_2}{\overset{S_1}{\gtrless}} \frac{P(s_2)}{P(s_1)} \quad / \ln$$

$$\log \frac{p(r|s_1)}{p(r|s_2)} \underset{S_2}{\overset{S_1}{\gtrless}} \log P(s_2) - \log P(s_1)$$

$$\frac{(r-\sqrt{E_b})^2 - (r+\sqrt{E_b})^2}{2\sigma_n^2} \underset{S_2}{\overset{S_1}{\gtrless}} \ln\frac{1-p}{p}$$

$$\sqrt{E_b}\, r \underset{S_2}{\overset{S_1}{\gtrless}} \frac{1}{2}\sigma_n^2 \ln\frac{1-p}{p} = \frac{1}{4}N_0 \ln\frac{1-p}{p}$$

need to know SNR

ML

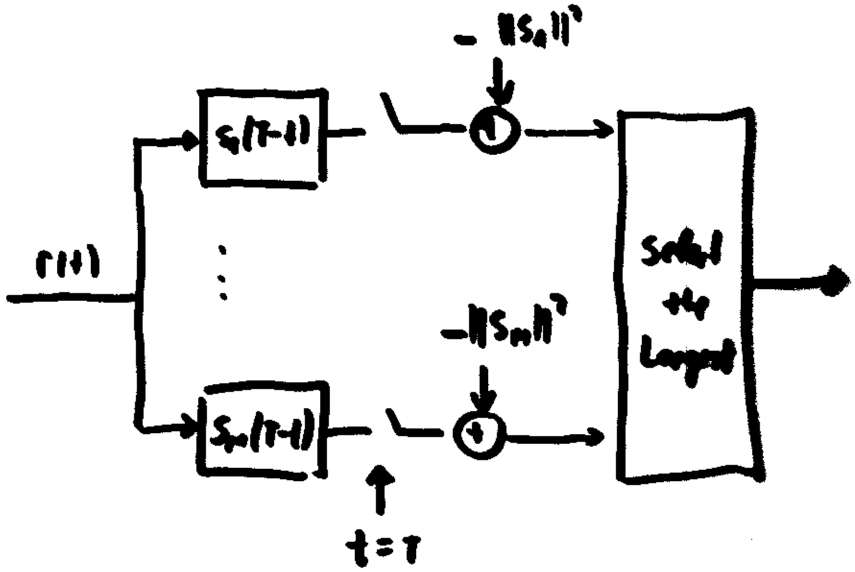
Equivalently we maximize $C(\bar{r}, \bar{s}_m) = 2\,\bar{r}\cdot\bar{s}_m - \|s_m\|^2 \quad (= -D'(\bar{r}, \bar{s}_m))$

correlation metrics

correlation term

energy term

PAM



$$C(\bar{r}, \bar{s}_m) = 2\int_0^T r(t)\, s_m(t)\,dt - \int_0^T s_m^2(t)\,dt$$

$$= 2\int_0^T \sum_{k=1}^{N} r_k \phi_k(t) \sum_{j=1}^{N} s_{mj}\,\phi_j(t) - \int_0^T \sum_{k=1}^{N} s_{m,k}\,\phi_k(t) \sum_{j=1}^{N} s_{m,j}\,\phi_j(t)\,dt$$

$$= 2\sum_{k=1}^{N} r_k \sum_{j=1}^{N} s_{m,j} \underbrace{\int_0^T \phi_k(t)\phi_j(t)\,dt}_{\delta_{k-j}} - \sum_{k=1}^{N} s_{m,k} \sum_{j=1}^{N} s_{m,j} \underbrace{\int_0^T \phi_k(t)\phi_j(t)\,dt}_{\delta_{k-j}}$$

$$= 2\sum_{k=1}^{N} r_k \cdot s_{m,k} - \sum_{k=1}^{N} s_{m,k}^2$$