Metric Spaces

Definition: A linear space X (over a fied F) is a metric space if there exist a nonnegative function $U \to \|U\|$ defined for every $U \in X$, called the norm such that:

1) $\|U\| = 0 \iff U = \theta$ θ - fixed point
2) $\|cu\| = |c| \cdot \|u\|$ - homogeneity
3) $\|u+v\| \leq \|U\| + \|V\|$ - triangle inequality

$u, v \in X, \; c \in K$

Definition: The metric is defined as $\rho(u,v) = \|U - V\|$

Properties:

$$\rho(u,v) = 0 \iff U = V$$
$$\rho(u,v) = \rho(v,u)$$
$$\rho(u,v) + \rho(v,w) \geq \rho(u,w)$$

Examples: $K = \mathbb{R}, \quad F = \mathbb{R}^n$

$$\|\vec{X}\|_p = \left(\sum_{k=1}^{n} |X_k|^p \right)^{\frac{1}{p}} \quad 1 \leq p < \infty$$

$$\|\vec{X}\|_\infty = \max_k |X_k|$$

$$\|\vec{X}\|_2 = \left(\sum_{k=1}^{n} |X_k|^2 \right)^{\frac{1}{2}} \text{ - Euclidian norm}$$

Example 2:

$$X = C[a,b] \qquad \|u\| = \max_{a \le t \le b} |u(t)| \qquad (1)$$

$$\|u\| = \int_a^b |u(t)|\,dt \qquad (2)$$

$$X = L^r(a,b) \qquad \int_a^b |u(t)|^r dt < \infty$$

$$\|u\| = \left(\int_a^b |u(t)|^r dt \right)^{\frac{1}{r}}$$

Definition: Let $\{u_n\}_{n \in N}$ be a sequence of points in the metric space X and let $u \in X$ be such that

$$\lim_{n \to \infty} \|u_n - u\| = 0$$

We say that u_n converges to u

Definition: A series $\{u_n\}_{n \in N}$ satisfying $\lim_{n,m \to \infty} \|u_n - u_m\| = 0$ is called the Cauchy series

Definition: Metric space is complete if every Cauchy series converges

Complete metric space is called the Banach space

Hilbert Space

Vector space over the complex field is called unitary (space with a scalar product) if there exists a function $(u,v) \in X^2 \to C$ satisfying :

1) $(u,u) \geq 0$
2) $(u,u) = 0 \quad 0 = 0$
3) $(u+v, w) = (u,w) + (v, w)$
4) $(cu, v) = c(u, v)$
5) $(u,v) = (v,u)^*$

The function (u,v) is a scalar product

Properties :

$$(u, cv) = c^*(u,v)$$
$$(u, v_1+v_2) = (u,v_1) + (u,v_2)$$
$$|(u,v)|^2 \leq (u,v)(v,v) \qquad \text{— Cauchy – Schwarz – ineq.}$$

Buniakowsky

Define a norm $\|u\| = \sqrt{(u,u)}$:

Unitary space with norm $\sqrt{(u,u)}$ is pre-Hilbert if it is a complete then it is called Hilbert space

Examples :

$\mathbb{R}^n$ is a Hilbert space $(\vec{x}, \vec{y}) = \sum_{k=1}^{n} x_k y_k = \vec{y}^T \cdot \vec{x}$

$\mathbb{C}^n$ is a Hilbert space $(\vec{x}, \vec{y}) = \sum_{k=1}^{n} x_k y_k^* = \vec{y}^* \cdot \vec{x}$

the norm: $\|x\| = \sqrt{(\vec{x}, \vec{x})} = \left(\sum_{k=1}^{n} |x_k|^2 \right)^{\frac{1}{2}}$ – Euclidian norm $\|x\|_E$

Buniakowsky – Cauchy – Schwartz – inequality

$$\left| \sum_{k=1}^{n} x_k y_k^* \right|^2 \leq \left(\sum_{k=1}^{n} |x_k|^2 \right) \cdot \left(\sum_{k=1}^{n} |y_k|^2 \right)$$

$L^2(a,b)$ is a Hilbert space $(u, v) = \int_a^b u(t)\, v^*(t)\, dt$

Orthogonal systems in Hilbert Spaces

Definition: A set of vectors $\{u_k\}_{k\in I}$ in the Hilbert space forms an orthogonal system if

$$(u_n, u_k) = \delta_{n,k} \|u_k\|^2 \qquad \forall n,k \in I$$

$$\delta_{n,k} = \begin{cases} 1 & n=k \\ 0 & n\neq k \end{cases} \qquad \|u_k\| = \sqrt{(u_k, u_k)}$$

Kroneckers delta

I - can be finite, countable or incountable

if $\|u_k\|=1$ the system is orthonormal

Gram-Schmidt Ortogonalization

Start from a set of countably many linearly independent vectors $\{v_0, v_1, \ldots, \}$. Gram-Schmidt procedure assignes an orthogonal system of vectors $\{u_0, u_1 \ldots\}$ to $\{v_0, v_1 \ldots\}$

$$u_0 = v_0$$

$$u_1 = v_1 + \lambda_{10} u_0 \qquad (u_1, u_0) = (v_1, u_0) + \lambda_{10} (u_0, u_0) = 0 \implies$$

$$\lambda_{10} = - \frac{(v_1, u_0)}{(u_0, u_0)}$$

Suppose that we have already constructed vectors $u_0, u_1, \ldots, u_{k-1}$

$$u_k = v_k + \lambda_{k0} u_0 + \lambda_{k1} u_1 + \ldots + \lambda_{k,k-1} u_{k-1}$$

$$(u_k, u_i) = (v_k, u_i) + \sum_{j=0}^{k-1} \lambda_{k,j} (u_j, u_i) = 0 \qquad \text{for } i = 0, 1, \ldots, k-1$$

$$\lambda_{k,i} = - \frac{(v_k, u_i)}{(u_i, u_i)}$$

$$u_k = v_k - \sum_{j=0}^{k-1} \frac{(v_k, u_i)}{(u_i, u_i)} u_i \qquad k = 1, 2, \ldots$$

Orthonormal basis: $\dfrac{u_k}{\|u_k\|}$

Example:

$v_1 = (1,1,1,1) \qquad v_2 = (1,2,4,5) \qquad v_3 = (1,-3,-4,-2)$

$u_1 = v_1 = (1,1,1,1)$

$$u_2 = v_2 - \frac{(v_2, u_1)u_1}{\|u_1\|} = (1,2,4,5) - \frac{(1+2+4+5)}{1^2+1^2+1^2+1^2}\cdot(1,1,1,1) = (-2,-1,1,2)$$

$$u_3 = v_3 - \frac{(v_3,u_1)}{\|u_1\|}u_1 - \frac{(v_3,u_2)}{\|u_2\|}u_2 = (1,-3,-4,-2) - \frac{(-8)}{4}(1,1,1,1) - \frac{(-7)}{10}(-2,-1,1,2)$$

$$= \left(\frac{8}{5}, -\frac{17}{10}, -\frac{13}{10}, \frac{7}{5}\right)$$

$u_1 = (1,1,1,1) \qquad \|u_1\| = 4$

$u_2 = (-2,-1,1,2) \qquad \|u_2\| = 10$

$u_3 = (16,-17,-13,14) \qquad \|u_3\| = 910$

$$u_1 = \frac{1}{2}(1,1,1,1)$$

$$u_2 = \frac{1}{\sqrt{10}}(-2,-1,1,2)$$

$$u_3 = \frac{1}{\sqrt{910}}(16,-17,-13,14)$$

Linear Spaces

Definition: A set X is a linear (vector) space over a field K if

1) A binary operation $+$ is defined on X so that $(X,+)$ form an Abelian group

2) For every pair (u,c) $u \in X, c \in K$ there exists an element of X so that

$$c_1(c_2 u) = (c_1 c_2) u$$
$$(c_1 + c_2) u = c_1 u + c_2 u$$
$$c(u_1 + u_2) = c_1 u_1 + c_2 u_2$$
$$1u = u$$

for every $u, u_i \in X$, $c, c_i \in K$ $i = 1, 2$.

elements of K – scalars
elements of X – vectors

Linearly independent vectors

Definition: Vectors u_i $1 \le i \le n$ from a linear vector space X are linearly independent if in the field K there are some scalars c_i $1 \le i \le n$ so that

$$\sum_{i=1}^{n} c_i u_i = \Theta$$

(Θ - is a zero element $c\Theta = \Theta$)

Definition: A linear space that has n linearly independ. vectors, and all sets of $n+1$ vectors are linearly dependent has the dimension n.

Theorem: Every vector from X can be obtained as a linear combination of a set of linearly independent vectors.

Definition: Set of linearly independent vectors forms a algebraic base

Base – coordinate system

Signal Space

- $L^2(a,b)$ set of finite energy functions $\int_a^b |u(t)|^2 dt < \infty$
- $L^2(a,b)$ is a Hilbert space
  - scalar product $(u,v) = \int_a^b u(t)v^*(t)dt$
  - norm $\|u\| = \sqrt{\int_a^b |u(t)|^2 dt}$

Hilbert Space - Banach space with norm $\sqrt{(u,u)} = \|u\|$
(unitary)

	Vector Space	Signal Space	Banach Space
Basis	$\vec{u} = \sum_{k=1}^{n} u_k \cdot \vec{b_k}$ $u_k = \vec{v} \cdot \vec{b_k}$	$u(t) = \sum_{k=1}^{\infty} u_k \phi_k(t)$ $u_k = (u(t), \phi_k(t))$	$u = \sum_{k=1}^{n} u_k b_k$ $u_k = (u_k, \vec{b_k})$
Scalar Product	$\vec{u} \cdot \vec{v} = \sum_{k=1}^{n} u_k v_k$	$(u(t), v(t)) = \int_a^b u(t) v^*(t) dt$	(u,v) – defined
Norm	$\lVert \vec{u} \rVert = \sqrt{\sum_{k=1}^{n} u_k^2}$	$\lVert u(t) \rVert = \sqrt{\int_a^b \lvert u(t) \rvert^2 dt}$	$\lVert u \rVert = (u, u)$
Metric (Distance)	$d_E(\vec{u}, \vec{v}) = \lVert \vec{u} - \vec{v} \rVert = \sqrt{\sum_{k=1}^{n} (u_k - v_k)^2}$	$d_E(u(t), v(t)) = \lVert u(t) - v(t) \rVert = \sqrt{\int_a^b \lvert u(t) - v(t) \rvert^2 dt}$	$\lVert u - v \rVert = \rho(u,v)$
Triangle Inequality	$\lVert \vec{u} + \vec{v} \rVert \leq \lVert \vec{u} \rVert + \lVert \vec{v} \rVert$	$\lVert u(t) + v(t) \rVert \leq \lVert u(t) \rVert + \lVert v(t) \rVert$	$\lVert u + v \rVert \leq \lVert u \rVert + \lVert v \rVert$
Schwartz Inequality	$\lvert \vec{u} \cdot \vec{v} \rvert \leq \lVert u \rVert \cdot \lVert v \rVert$	$\lvert (u(t), v(t)) \rvert \leq \lVert u(t) \rVert \cdot \lVert v(t) \rVert$	$\lvert (u,v) \rvert \leq \lVert u \rVert \cdot \lVert v \rVert$

Basis Functions:

Assume set of ortonormal functions $\{\phi_k(t),\ k=1,2,\dots\}$ in $L^2[a,b]$

$$(\phi_i(t), \phi_j(t)) = \int_a^b \phi_i(t)\,\phi_j^*(t)\,dt = \delta_{ij}$$

Aproximate $u(t)$ by K orthonormal functions

$$u(t) \approx U_K(t) = \sum_{k=1}^{K} u_k\,\phi_k(t)$$

Find u_k to minimize $\| u(t) - U_K(t)\|^2 = \varepsilon$ - error

$$\varepsilon = \int_a^b |u(t) - U_K(t)|^2\,dt = \int_a^b \left| u(t) - \sum_{k=1}^{n} u_k\,\phi_k(t)\right|^2 dt$$

$$\frac{\partial \varepsilon}{\partial u_n} = \int_a^b \left(u(t) - \sum_{k=1}^{K} u_k\,\phi_k(t)\right)\phi_n^*(t)\,dt = 0$$

$$\int_a^b \Big(u(t)\,\phi_n^*(t) - \underbrace{\sum_{k=1}^{K} u_k\,\phi_k(t)\,\phi_n^*(t)}_{u_n}\Big)\,dt = 0$$

$$u_n = \int_a^b (u(t), \phi_n^*(t))\,dt = (u(t), \phi_n(t))$$

Complete Ortonormal Sets and Generalized Furier Coeff.

Hilbert space: complete orthonormal set $\{u_k,\ k=1,2,\dots\}$

$$u = \sum a_k u_k \qquad a_k - \text{Fourier coefficients}$$

$L^2(a,b)$: $\qquad \mathcal{E}_{min} = 0 \qquad u(t) = \sum_{k=1}^{K} u_k \phi_k(t)$

A set or orthonormal functions in $L^2(a,b)$ is called complete orthonormal set if $\mathcal{E}_{min} = 0$

Example $\quad [a,b] = [-\frac{T}{2}, \frac{T}{2}) \quad - \ L^2(a,b)$ - Lipshitz condit.

$$\phi_k(t) = \frac{1}{T} e^{+j2\pi k \frac{t}{T}} \qquad k = 0, 1, \dots$$

$$u_k = \int_{-T/2}^{T/2} u(t)\, \phi_k^*(t)\, dt$$

$$u_k = \frac{1}{T} \int_{-T/2}^{T/2} u(t)\, e^{-j 2\pi k \frac{t}{T}}\, dt$$

complete orthonormal set