Signal Space

Consider complex signals defined on some time interval $[a, b]$.
The inner product of signals $x(t)$ and $y(t)$ is defined as

$$(x(t), y(t)) = \int_a^b x(t)\, y^*(t)\, dt$$

sometimes we write $\langle x(t), y(t) \rangle$

The norm of a signal $x(t)$

$$\| x(t) \| = \left(\int_a^b |x(t)|^2 dt \right)^{\frac{1}{2}}$$

note $\| x(t) \|^2 = (x(t), x(t))$

Triangle inequality: $\| x(t) + y(t) \| = \| x(t) \| + \| y(t) \|$

Cauchy-Schwarz-Bunjakovsky inequality:

$$\left| \int_a^b x(t)\, y^*(t)\, dt \right| \le \left(\int_a^b |x(t)|^2 dt \right)^{\frac{1}{2}} \cdot \left(\int_a^b |y(t)|^2 dt \right)^{\frac{1}{2}}$$

Orthogonal expansion of signals

Let $s(t)$ be deterministic real-valued signal with finite energy

$$\mathcal{E}_s = \int_{-\infty}^{+\infty} |s(t)|^2 dt$$

Let $\{ \phi_1(t), \phi_2(t), \ldots, \phi_K(t) \}$ be a set of orthonormal functions, i.e

$$\int_{-\infty}^{+\infty} \phi_i(t)\, \phi_j(t)\, dt = \delta_{i,j}$$

$\delta_{i,j}$ - Kronecker delta

$s(t)$ can be approximated by

$$\hat{s}(t) = \sum_{k=1}^{K} s_k\, \phi_k(t)$$

We choose s_k to minimize the energy of the approximation error signal

$$e(t) = s(t) - \hat{s}(t)$$

$$\mathcal{E}_e = \int_{-\infty}^{+\infty} |s(t) - \hat{s}(t)|^2 dt$$

$$\mathcal{E}_e = \int_{-\infty}^{+\infty} \left(s(t) - \sum_{k=1}^{K} s_k \phi_k(t) \right)^2 dt$$

Optimum coefficients:

1) - take derivatives of $\mathcal{E}_e$ with respect to $s_1, s_2, \dots, s_K$ and set the derivatives to 0

2) - use the result from estimation theory mean-square criterion.

The minimum of $\mathcal{E}_e$ with respect to $s_1, s_2, \dots, s_K$ is obtained when the error is orthogonal to each of the functions $\phi_k(t)$ $k=1,\dots,K$

$$\int_{-\infty}^{+\infty} \underbrace{\left(s(t) - \sum_{k=1}^{K} s_k \phi_k(t) \right)}_{e(t)} \cdot \phi_n(t)\, dt = 0 \qquad n = 1, 2, \dots K$$

$$\Rightarrow \int_{-\infty}^{+\infty} s(t)\phi_n(t)\, dt - \sum_{k=1}^{K} s_k \underbrace{\int_{-\infty}^{+\infty} \phi_k(t)\phi_n(t)\, dt}_{\delta_{k,n}} = 0$$

$\phi_k(t), \phi_n(t)$ - orthonormal

$$\Rightarrow \int_{-\infty}^{+\infty} s(t)\phi_n(t)\, dt = \sum_{k=1}^{K} s_k \delta_{k,n} = s_n$$

$$\Rightarrow s_n = \int_{-\infty}^{+\infty} s(t)\phi_n(t)\, dt \qquad n = 1, 2, \dots, K$$

- $\hat{s}(t)$ is projection of $s(t)$ onto the K-dimensional signal space spanned by $\phi_k(t)$!

Minimum approximation error (in the mean square sense)

$$\mathcal{E}_{e\,min} = \int_{-\infty}^{+\infty} e(t)\, e(t)\, dt$$

$$\mathcal{E}_{e_{min}} = \int_{-\infty}^{+\infty} \left(s(t) - \sum_{k=1}^{K} s_k \phi_k(t)\right)\left(s(t) - \sum_{m=1} s_m \phi_m(t)\right) dt$$

$$= \int_{-\infty}^{+\infty} s^2(t)\, dt - 2 \sum_{k=1}^{K} s_k \int_{-\infty}^{+\infty} s(t)\, \phi_k(t) + \sum_{k=1}^{K} \sum_{m=1}^{K} s_m s_k \underbrace{\phi_k(t) \phi_m(t)}_{\text{orthogonal}}$$

two identical term.

$$= \underbrace{\int s^2(t)\, dt}_{\mathcal{E}_s} - 2 \sum_{k=1}^{K} s_k^2 + \sum_{k=1}^{K} s_k^2$$

$$= \mathcal{E}_s - \sum_{k=1}^{K} s_k^2$$

When $\mathcal{E}_{e_{min}} = 0 \Rightarrow \sum_{k=1}^{K} s_k^2 = \mathcal{E}_s = \int_{-\infty}^{+\infty} s^2(t)\, dt$

$$s(t) = \sum_{k=1}^{K} s_k \phi_k(t)$$

This can be extended to complex-valued signals.

Example:

$$\left.\begin{aligned} \phi_1(t) &= \sqrt{\tfrac{2}{T}} \cos w_c t \\ \phi_2(t) &= -\sqrt{\tfrac{2}{T}} \sin w_c t \end{aligned}\right\} \text{orthonormal on } [0, T]$$

$t \in [0, T]$