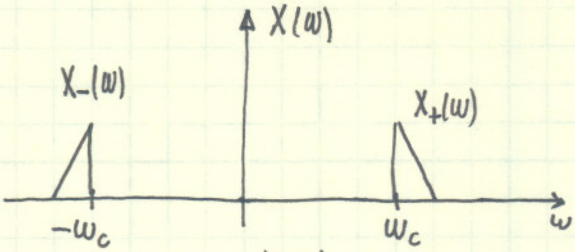
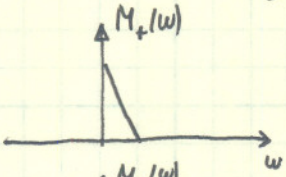
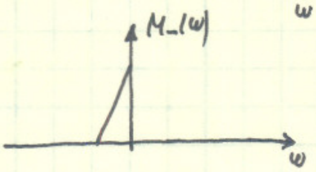
Representation of Band-Pass Signals







$$X_+(\omega) = M_+(\omega - \omega_c)$$
$$X_-(\omega) = M_-(\omega + \omega_c)$$

$$X(\omega) = X_-(\omega) + X_+(\omega)$$

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\}$$
$$x_-(t) = \mathcal{F}^{-1}\{X_-(\omega)\}$$
$$x_+(t) = \mathcal{F}^{-1}\{X_+(\omega)\}$$

$$x(t) = x_-(t) + x_+(t)$$
$$x_+(t) = m_+(t)\, e^{j\omega_c t}$$
$$x_-(t) = m_-(t)\, e^{-j\omega_c t}$$
$$m_-(t) = \mathcal{F}^{-1}\{M_-(\omega)\}$$
$$m_+(t) = \mathcal{F}^{-1}\{M_+(\omega)\}$$

- $m_+(t)$ and $m_-(t)$ cannot be real since their spectra are not symmetric
- but $m_-(t) + m_+(t)$ – is real.

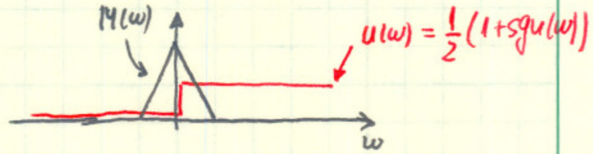
$$m(t) = m_-(t) + m_+(t)$$

$$M(\omega) = M_-(\omega) + M(\omega) \quad \text{– symmetric}$$

$$m_-(t) = \frac{1}{2} m(t) - \frac{1}{2} j\hat{m}(t)$$
$$m_+(t) = \underset{\text{real}}{\frac{1}{2} m(t)} + \frac{1}{2} j\underset{\text{real}}{\hat{m}(t)}$$

what is $\hat{m}(t)$?

$$M_+(\omega) = M(\omega) \cdot \underset{\text{unit step function}}{U(\omega)}$$



$$M_+(\omega) = M(\omega) \cdot \frac{1}{2}(1 + \mathrm{sgn}(\omega))$$
$$= \frac{1}{2} M(\omega) + \frac{1}{2} M(\omega)\,\mathrm{sgn}(\omega)$$

since $m_+(t) = \frac{1}{2} m(t) + \frac{1}{2} j\hat{m}(t)$ ($\mathcal{F}$, $\mathcal{F}^{-1}$, $\mathcal{F}^{-1}$)

$$\hat{m}(t) = \mathcal{F}^{-1}\{\underset{\frac{\pi}{2}\text{ - shift}}{-jM(\omega)\cdot \mathrm{sgn}(\omega)}\}$$

$$\hat{m}(t) = m(t) * \mathcal{F}^{-1}\{-j\,\mathrm{sgn}(\omega)\}$$

for every x: $\mathcal{F}\{x(t)\} = X(\omega) \Rightarrow \frac{1}{2\pi}\mathcal{F}\{X(t)\} = x(-\omega)$

$$\mathcal{F}\{\mathrm{sgn}(t)\} = \frac{2}{j\omega} \Rightarrow$$
$$\Rightarrow \frac{1}{2\pi}\mathcal{F}\left\{\frac{2}{jt}\right\} = \mathrm{sgn}(-\omega)$$
$$\mathcal{F}\left\{\frac{1}{\pi t}\right\} = j\,\mathrm{sgn}(-\omega) = -j\,\mathrm{sgn}(\omega)$$

$$\Rightarrow \hat{m}(t) = m(t) * \frac{1}{\pi t}$$

$$\hat{m}(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{m(\tau)}{t - \tau}\, d\tau$$

Hilbert transform

Hilbert transformer

$$x(t) = x_-(t) + x_+(t)$$
$$= m_-(t)\, e^{-j\omega_c t} + m_+(t)\, e^{j\omega_c t}$$
$$= \left(\tfrac{1}{2} m(t) - \tfrac{1}{2} j \hat{m}(t)\right) e^{-j\omega_c t}$$
$$+ \left(\tfrac{1}{2} m(t) + \tfrac{1}{2} j \hat{m}(t)\right) e^{j\omega_c t}$$
$$= m(t)\, \tfrac{1}{2}\left(e^{-j\omega_c t} + e^{j\omega_c t}\right)$$
$$- \hat{m}(t)\, \tfrac{1}{2j}\left(e^{j\omega_c t} - e^{-j\omega_c t}\right)$$

in phase, quadrature

$$\Rightarrow x(t) = m(t)\cos\omega_c t - \hat{m}(t)\sin\omega_c t$$

Consider now:

$$(m(t) + j\hat{m}(t))(\cos\omega_c t + j\sin\omega_c t)$$
$$= m(t)\cos\omega_c t - \hat{m}(t)\sin\omega_c t \quad \text{— real part}$$
$$+ j(\hat{m}(t)\cos\omega_c t + m(t)\sin\omega_c t) \quad \text{— imaginary}$$

$$\Rightarrow x(t) = \mathrm{Re}\left\{(m(t) + j\hat{m}(t))\, e^{j\omega_c t}\right\}$$

$$x(t) = \mathrm{Re}\left\{\tilde{x}(t)\, e^{j\omega_c t}\right\}$$

complex envelope

analytic signal

$$\tilde{x}(t) = m(t) + j\hat{m}(t) = 2m_+(t)$$

envelope

phase

$$\tilde{x}(t) = a(t)\, e^{j\theta(t)}$$

$$x(t) = \mathrm{Re}\left\{a(t)\, e^{j(\omega_c t + \theta(\omega))}\right\}$$

$$x(t) = a(t)\cos(\omega_c t + \theta(\omega))$$

instead

we consider: $a(t)\, e^{j\theta(\omega)}$

$\mathrm{Re}\{z\} = \frac{1}{2}(z + z^*)$

$$X(\omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\left\{\mathrm{Re}\left\{\tilde{x}(t)\, e^{j\omega_c t}\right\}\right\}$$
$$= \mathcal{F}\left\{\tfrac{1}{2}\tilde{x}(t)\, e^{j\omega_c t} + \tfrac{1}{2}\tilde{x}^*(t)\, e^{-j\omega_c t}\right\}$$
$$= \tfrac{1}{2}\tilde{X}(\omega - \omega_c) + \tfrac{1}{2}\tilde{X}^*(-(\omega + \omega_c)) \quad \text{--- } \Delta$$

$$x(t) = a(t)\cos(\omega_c t + \theta(\omega))$$

$$\overline{x^2(t)} = \tfrac{1}{2}\overline{a^2(t)} \quad \text{— energies}$$

$$\mathcal{E} = \tfrac{1}{2}\int_{-\infty}^{+\infty} |\tilde{x}(t)|\, dt.$$

↑ half a energy of complex envelope.

④

$$x(t) = a(t) + jb(t)$$

$$X(w) = A(w) + jB(w) = A_R(w) + jA_I(w) + j(B_R(w) + jB_I(w))$$

$$= A_R(w) - B_I(w) + j(A_I(w) + B_R(w))$$

$$X(-w) = A_R(-w) - B_I(-w) + j(A_I(-w) + B_R(-w))$$

$$= A_R(w) + B_I(w) + j(-A_I(w) + B_R(w))$$

$$= A_R(w) + B_I(w) - j(A_I(w) - B_R(w))$$

$$X^*(-w) = A_R(w) + B_I(w) + j(A_I(w) - B_R(w))$$

$F_R(-w) = F_R(w)$

$F_I(-w) = -F_I(w)$

□

$$x^*(t) = a(t) - jb(t)$$

$$\mathcal{F}\{x^*(t)\} = A(w) - jB(w) = A_R(w) + jA_I(w) - j(B_R(w) + jB_I(w))$$

$$= A_R(w) + B_I(w) + j(A_I(w) - B_R(w))$$

$$= X^*(-w)$$

$$F(w) = F_R(w) + jF_I(w)$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(w)e^{jwt}\,dw = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(F_R(w) + jF_I(w))(\cos wt + j\sin wt)\,dw$$

real

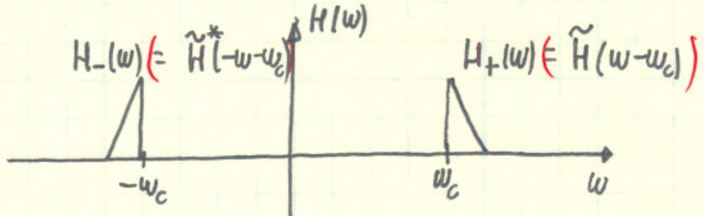
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}(F_R(w)\cos wt - F_I(w)\sin wt)\,dw$$

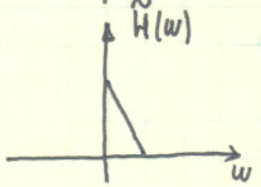
$$+ \frac{j}{2\pi}\underbrace{\int_{-\infty}^{+\infty}(F_I(w)\cos wt + F_R(w)\sin wt)\,dw}_{=0 \text{ since } f(t) \text{ is real}}$$

$\Rightarrow$ $F_R(w)$ must be an even function and $F_I(w)$ must be an odd function of w. $\Rightarrow$ $F_R(-w) = F_R(w)$

$F_I(-w) = -F_I(w)$

Band-Pass Systems



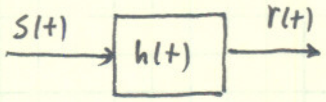


$$H(w) = \tilde{H}(w-w_c) + \tilde{H}^*(-w-w_c)$$

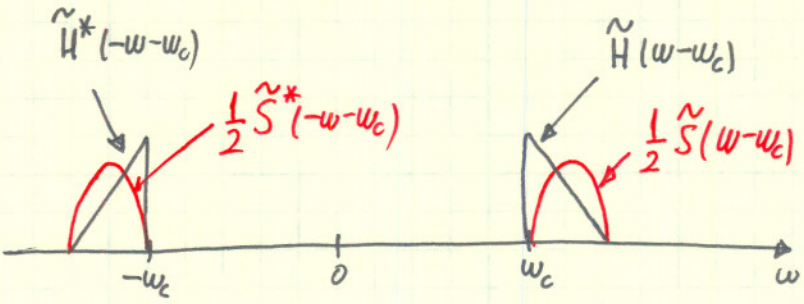
$$h(t) = \tilde{h}(t)\, e^{jw_c t} + \tilde{h}^*(t)\, e^{-jw_c t}$$

$$h(t) = 2\, \mathrm{Re}\{\tilde{h}(t)\, e^{jw_c t}\}$$

notice the factor two (non-existant in def. of complex envelope of ~~the~~ signal)



$$R(\omega) = S(\omega)\cdot H(\omega)$$



$$R(\omega) = \frac{1}{2}\underbrace{\tilde{H}(\omega-\omega_c)\cdot\tilde{S}(\omega-\omega_c)}_{\tilde{R}(\omega-\omega_c)} + \frac{1}{2}\underbrace{\tilde{H}^*(-\omega-\omega_c)\tilde{S}^*(-\omega-\omega_c)}_{\tilde{R}^*(-\omega-\omega_c)}$$

$$R(\omega) = \frac{1}{2}\tilde{R}(\omega-\omega_c) + \frac{1}{2}\tilde{R}^*(-\omega-\omega_c)$$

$$\Rightarrow \quad \tilde{R}(\omega) = \tilde{S}(\omega)\cdot\tilde{H}(\omega).$$

$$R(\omega) = S(\omega)H(\omega)$$

$$R(\omega) = \frac{1}{2}\left(\tilde{S}(\omega-\omega_c) + \tilde{S}^*(-\omega-\omega_c)\right)\cdot\left(\tilde{H}(\omega-\omega_c) + \tilde{H}^*(-\omega-\omega_c)\right)$$

$$= \frac{1}{2}\tilde{S}(\omega-\omega_c)\cdot\tilde{H}(\omega-\omega_c) + \frac{1}{2}\tilde{S}^*(-\omega-\omega_c)\cdot\tilde{H}^*(-\omega-\omega_c)$$

$$+ \frac{1}{2}\underbrace{\tilde{S}(\omega-\omega_c)\cdot\tilde{H}^*}_{\substack{\text{don't overlap} \\ \text{product}=0}}(-\omega-\omega_c) + \frac{1}{2}\underbrace{\tilde{S}^*(-\omega-\omega_c)}_{\substack{\text{don't overlap} \\ \text{product}=0}}\tilde{H}(\omega-\omega_c)$$

$$R(\omega) = \frac{1}{2}\tilde{S}(\omega-\omega_c)\cdot\tilde{H}(\omega-\omega_c) + \frac{1}{2}S^*(-\omega-\omega_c)\cdot\tilde{H}^*(-\omega-\omega_c)$$