4.5 BANDLIMITED PROCESSES AND SAMPLING

A bandlimited process is a random process whose power-spectral density occupies a finite bandwidth. In other words, a bandlimited process is a process with the property that for all $|f| \geq W$, we have $\mathcal{S}_X(f) \equiv 0$ where W is the bandwidth of the process.

Almost all of the processes encountered in nature are bandlimited because there is a limit on the bandwidth of all physical systems. Bandlimited processes also arise when passing random processes through bandlimited linear systems. The output, usually called a "filtered process," is a bandlimited process. The power-spectral density of a typical bandlimited process is shown in Figure 4.23.

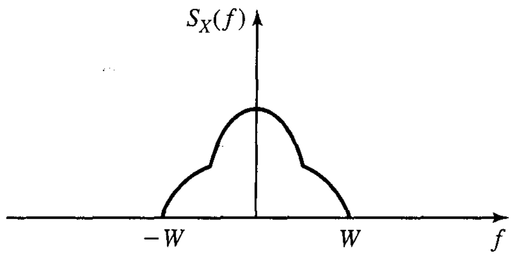

Figure 4.23 Power spectrum of a bandlimited process.

We have already seen in Section 2.4 that for bandlimited signals there exists the powerful sampling theorem which states that the signal can be perfectly reconstructed from its sampled values as long as the sampling rate is more than twice the highest frequency component in the signal; i.e., the bandlimited signal can be expressed in terms of its samples taken at regular intervals T_s, where $T_s \leq \frac{1}{2W}$, by the relation

$$x(t) = \sum_{k=-\infty}^{\infty} 2WT_s \, x(kT_s) \, \text{sinc}(2W(t - kT_s))$$

In the special case where $T_s = \frac{1}{2W}$, the above relation simplifies to

$$x(t) = \sum_{k=-\infty}^{\infty} x\left(\frac{k}{2W}\right) \text{sinc}\left(2W\left(t - \frac{k}{2W}\right)\right)$$

One wonders if such a relation exists for bandlimited random processes. In other words, if it is possible to express a bandlimited process in terms of its sampled values. The following theorem, which is the sampling theorem for the bandlimited random processes, shows that this is in fact true.

Theorem 4.5.1. Let $X(t)$ be a stationary bandlimited process; i.e., $\mathcal{S}_X(f) \equiv 0$ for $|f| \geq W$. Then the following relation holds

$$E\left|X(t) - \sum_{k=-\infty}^{\infty} X(kT_s) \, \text{sinc}(2W(t - kT_s))\right|^2 = 0 \tag{4.5.1}$$

where $T_s = \frac{1}{2W}$ denotes the sampling interval.

Proof. Let us start by expanding the above relation. The left-hand side becomes

$$E\left|X(t)-\sum_{k=-\infty}^{\infty} X(kT_s)\operatorname{sinc}(2W(t-kT_s))\right|^2$$

$$= R_X(0) - 2\sum_{k=-\infty}^{\infty} R_X(t-kT_s)\operatorname{sinc}(2W(t-kT_s))$$

$$+ \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} R_X((k-l)T_s)\operatorname{sinc}(2W(t-kT_s))\operatorname{sinc}(2W(t-lT_s))$$

Introducing the change of variable $m = l-k$ in the last line of the above relation, we have

$$\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} R_X((k-l)T_s)\operatorname{sinc}(2W(t-kT_s))\operatorname{sinc}(2W(t-lT_s))$$

$$= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_X(-mT_s)\operatorname{sinc}(2W(t-kT_s))\operatorname{sinc}(2W(t-kT_s-mT_s))$$

$$= \sum_{k=-\infty}^{\infty}\operatorname{sinc}(2W(t-kT_s))\sum_{m=-\infty}^{\infty} R_X(mT_s)\operatorname{sinc}(2W(t-kT_s-mT_s))$$

where we have used the fact that $R_X(-mT_s) = R_X(mT_s)$.

The fact that the process is bandlimited means that the Fourier transform of $R_X(\tau)$ is bandlimited to W and, therefore, for $R_X(\tau)$ we have the expansion

$$R_X(t) = \sum_{k=-\infty}^{\infty} R_X(kT_s)\operatorname{sinc}[2W(t-kT_s)]$$

from which we obtain

$$\sum_{m=-\infty}^{\infty} R_X(mT_s)\operatorname{sinc}[2W(t-kT_s-mT_s)] = R_X(t-kT_s)$$

Therefore,

$$E\left|X(t)-\sum_{k=-\infty}^{\infty} X(kT_s)\operatorname{sinc}[2W(t-kT_s)]\right|^2$$

$$= R_X(0) - 2\sum_{k=-\infty}^{\infty} R_X(t-kT_s)\operatorname{sinc}[2W(t-kT_s)]$$

$$+ \sum_{k=-\infty}^{\infty} R_X(t-kT_s)\operatorname{sinc}[2W(t-kT_s)]$$

$$= R_X(0) - \sum_{k=-\infty}^{\infty} R_X(t-kT_s)\operatorname{sinc}[2W(t-kT_s)]$$

Now we can apply the result of Problem 2.42 to $R_X(\tau)$ to obtain

$$R_X(0) = \sum_{k=-\infty}^{\infty} R_X(t - kT_s)\,\mathrm{sinc}[2W(t - kT_s)]$$

Substituting this result in the expression for $E[X(t)]$, we obtain

$$E\left|X(t) - \sum_{k=-\infty}^{\infty} X(kT_s)\,\mathrm{sinc}[2W(t - kT_s)]\right|^2 = 0$$

This concludes the proof of the theorem. ■

This result is a parallel to the sampling theorem for deterministic signals developed in Section 2.4. Note that, due to the random nature of the entities involved in this case, the equality of $X(t)$ and $\sum_{k=-\infty}^{\infty} X(kT_s)\,\mathrm{sinc}[2W(t - kT_s)]$ is not pointwise, and we can only say that

$$E\left|X(t) - \sum_{k=-\infty}^{\infty} X(kT_s)\,\mathrm{sinc}[2W(t - kT_s)]\right|^2 = 0$$

This is usually called *equality in quadratic mean* or *equality in the mean-squared sense* and denoted by

$$X(t) \overset{\text{q.m.}}{=} \sum_{k=-\infty}^{\infty} X(kT_s)\,\mathrm{sinc}[2W(t - kT_s)] \tag{4.5.2}$$

Now that we have seen that a bandlimited process can be recovered from its sampled values taken at $1/2W$ intervals, an interesting question is whether or not these samples are uncorrelated. It can be shown that a necessary and sufficient condition for the uncorrelatedness of these samples is that the power spectrum of the process be flat over the passband; i.e.

$$\mathcal{S}_x(f) = \begin{cases} K & |f| < W \\ 0, & \text{otherwise} \end{cases}$$

Unless this condition is satisfied, the samples of the process will be correlated and this correlation can be exploited to make their transmission easier. This fact will be further explored in Chapter 6.