Example Symmetric channels

- Each row of Π contains the same set of probabilities $\{p_j\}_{1\le j\le n}$ ①
and each column contains the same set of numbers $\{q_i\}_{1\le i\le m}$ ⊛

- Examples:

$$\Pi = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \end{bmatrix} \qquad \Pi = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

- An important property of the symmetric channel is that $H(Y|X)$ is independent of $P(X_i)$ and depends only on the channel transition probability matrix.

$$H(Y|X) = \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i)\, P(y_j|x_i) \log \frac{1}{P(y_j|x_i)}$$

$$= \sum_{i=1}^{m} P(x_i) \underbrace{\sum_{j=1}^{n} P(y_j|x_i) \log \frac{1}{P(y_j|x_i)}}_{\text{independent on } i}$$

$$= \underbrace{\sum_{i=1}^{m} P(x_i)}_{=1} \underbrace{\sum_{j=1}^{n} p_j \log \frac{1}{p_j}}_{\text{still independent on } i}$$

$$= \sum_{j=1}^{n} p_j \log \frac{1}{p_j}$$

- We have proved that the conditional entropy $H(Y|X)$ does not depend on the input probability distribution. Thus the problem of maximizing $I(X;Y) = H(Y) - H(Y|X)$ reduces to the problem of maximizing the output entropy $H(Y)$.

- We know that $H(Y) \leq \log_2 n$ where the equality is achieved when

$$P(y_j) = \frac{1}{n} \qquad 1 \leq j \leq n$$

- We prove that the output symbols are equally likely when so are input symbol

$$P(y_j) = \sum_{i=1}^{m} P(x_i) P_{ij} = \sum_{i=1}^{m} P(x_i)\, q_i \quad \Rightarrow \quad \text{if } P(x_i) = \frac{1}{m}$$

$\uparrow$ (*) independent on j.

$$P(y_j) = \frac{1}{m} \sum_{i=1}^{m} q_i \text{ — all equal}$$

- Thus all symbols $y \in Y$ have the same probability and the capacity of a symmetric channel is given by

$$C = \log n + \sum_{j=1}^{n} P_j \log_2 P_j$$

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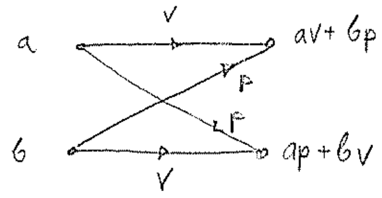
Example

Binary Symetric Channel (BSC)



$$I(X;Y) = H(Y) - H(Y|X)$$

$$= H(Y) - \left(a \cdot v \log \frac{1}{v} + a \cdot p \log \frac{1}{p} + b\, p \log \frac{1}{p} + b\, v \log \frac{1}{v}\right)$$

$$= H(Y) - (a+b)\Big(\underbrace{p \log \frac{1}{p} + v \log \frac{1}{v}}_{H(p)}\Big)$$

$$= H(ap+bv) - H(p)$$

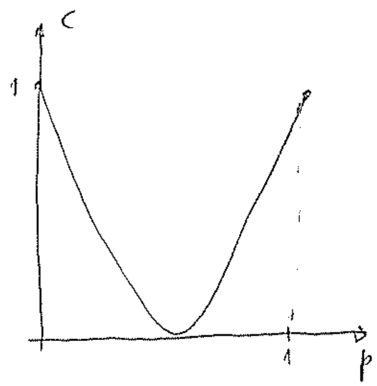
$$H(x) = x \log \frac{1}{x} + (1-x) \log \frac{1}{1-x}$$

$$\left.\begin{array}{ll} p \le ap+bv \le v & \text{for } p \le \frac{1}{2} \\ p \ge ap+bv \ge v & \text{for } p \ge \frac{1}{2} \end{array}\right\} \Rightarrow H(ap+bv) \ge H(p)$$

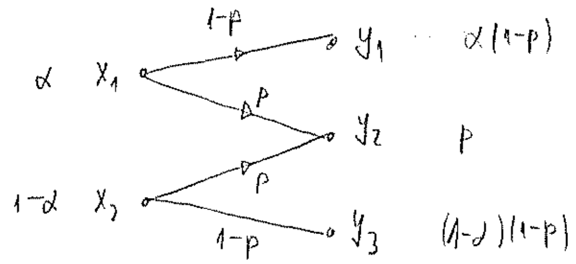
$$\Rightarrow I(X;Y) \ge 0$$

For a fixed p $H(ap+bv)$ achieves maximum for $a = \frac{1}{2}$ $H(ap+bv) = H(\frac{1}{2}p + \frac{1}{2}(1-p)) = H(\frac{1}{2}) = 1$ $\frac{\text{bit}}{\text{symb}}$

$$I_{max}(X;Y) = 1 - H(p) \qquad C = 1 - H(p)$$



The binary erasure channel (BEC)



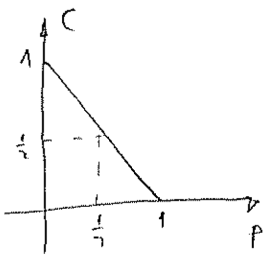
$$I(X;Y) = H(Y) - H(Y|X)$$

$$= -\alpha(1-p)\log\alpha(1-p) - p\log p - (1-\alpha)(1-p)\log(1-\alpha)(1-p)$$

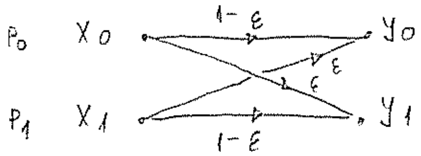
$$+\ \alpha(1-p)\log(1-p) + \alpha p\log p + (1-\alpha)p\log p + (1-\alpha)(1-p)\log(1-p)$$

$\Rightarrow$

$$C = \max_{p(x_i)} I(X;Y) = I(X;Y)\Big|_{\alpha=\frac{1}{2}} = 1-p$$



Binary Symmetric Channel



$$I(X,Y) = H(Y) - H(Y|X)$$

$$P(y_0) = P(X_0)\cdot(1-\varepsilon) + P(X_1)\cdot\varepsilon = P(X_0) + (P(X_0)-P(X_1))\varepsilon$$

$$P(y_1) = P(X_1) + (P(X_1)-P(X_0))\varepsilon$$

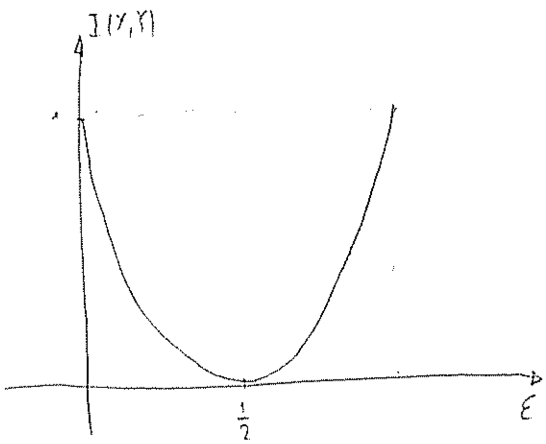
$$H(Y|X) = \sum_{i=0}^{1}\sum_{j=0}^{1} P(x_i, y_j)\log\frac{1}{P(y_j|x_i)} = \sum_{i=0}^{1}\sum_{j=0}^{1} P(x_i)\, P(y_j|x_i)\log\frac{1}{P(y_j|x_i)}$$

$$H(Y|X) = P_0(1-\varepsilon)\log\frac{1}{1-\varepsilon} + P_0\,\varepsilon\log\frac{1}{\varepsilon} + P_1\,\varepsilon\log\frac{1}{\varepsilon} + P_1(1-\varepsilon)\log\frac{1}{1-\varepsilon}$$

$P_0 = P_1 = \frac{1}{2}$ $\qquad H(Y|X) = (1-\varepsilon)\log\frac{1}{1-\varepsilon} + \varepsilon\log\frac{1}{\varepsilon} = H(\varepsilon)$

$\qquad H(Y) = 1$

$$I(X,Y) = 1 - H(\varepsilon)$$



How to chose input probabilities to maximize $I(X,Y)$
(DMC)

$$I(x_i;Y) = \sum_{j=1}^{n} P(y_j|x_i) \log \frac{P(y_j|x_i)}{P(y_j)}$$

$$I(X;Y) = \sum_{i=1}^{m} P(x_i)\, I(y_j|x_i)$$

$$I(x_i;Y) = \begin{cases} = C & P(x_i) > 0 \\ \leq C & P(x_i) = 0 \end{cases}$$

Arimoto-Blahut algorithm