Example 4.3.8

Let $X(t)$ represent the process in Example 4.2.3, and let $Z(t) = X(t) + \frac{d}{dt}X(t)$. Then,

$$\mathcal{S}_{XY}(f) = \frac{jA^2\pi f_0}{2}\left[\delta(f+f_0) - \delta(f-f_0)\right]$$

and, therefore,

$$\mathrm{Re}[\mathcal{S}_{XY}(f)] = 0$$

Hence,

$$\mathcal{S}_Z(f) = \mathcal{S}_X(f) + \mathcal{S}_Y(f) = A^2\left(\tfrac{1}{4} + \pi^2 f_0^2\right)[\delta(f-f_0) + \delta(f+f_0)]$$

4.4 GAUSSIAN AND WHITE PROCESSES

Gaussian processes play an important role in communication systems. The fundamental reason for their importance is that thermal noise in electronic devices, which is produced by random movement of electrons due to thermal agitation, can be closely modeled by a Gaussian process. The reason for the Gaussian behavior of thermal noise is that the current introduced by movement of electrons in an electric circuit can be regarded as the sum of small currents of a very large number of sources, namely individual electrons. It can be assumed that at least a majority of these sources behave independently and, therefore, the total current is the sum of a large number of i.i.d. random variables. Now by applying the central limit theorem, this total current has a Gaussian distribution.

Apart from thermal noise, Gaussian processes provide rather good models for some information sources as well. Some interesting properties of the Gaussian processes, which will be discussed in this section, make these processes mathematically tractable and easy to deal with.

4.4.1 Gaussian Processes

We start our discussion with a formal definition of Gaussian processes.

Definition 4.4.1. A random process $X(t)$ is a *Gaussian process* if for all n and all $(t_1, t_2, \ldots, t_n)$, the random variables $\{X(t_i)\}_{i=1}^n$ have a jointly Gaussian density function.

From the above definition it is seen that, in particular, at any time instant t_0 the random variable $X(t_0)$ is Gaussian, and at any two points t_1, t_2 random variables $(X(t_1), X(t_2))$ are distributed according to a two-dimensional Gaussian random variable. Moreover, since a complete statistical description of $\{X(t_i)\}_{i=1}^n$ depends only on $\mathbf{m}$ and $\mathbf{C}$, the mean and autocovariance matrices, we have the following theorem.

Theorem 4.4.1. For Gaussian processes, knowledge of the mean and autocorrelation; i.e., $m_X(t)$ and $R_X(t_1, t_2)$ gives a complete statistical description of the process. ■

The following theorem is of fundamental importance in dealing with Gaussian processes.

Theorem 4.4.2. If the Gaussian process $X(t)$ is passed through an LTI system, then the output process $Y(t)$ will also be a Gaussian process.

Proof. To prove that $Y(t)$ is Gaussian we have to prove that for all n and all $\{t_i\}_{i=1}^n$, the vector $(Y(t_1), Y(t_2), \ldots, Y(t_n))$ is a Gaussian vector. In general, we have

$$Y(t_i) = \int_{-\infty}^{\infty} X(\tau)h(t_i - \tau)\,d\tau = \lim_{N\to\infty}\lim_{\Delta\to 0}\sum_{j=-N}^{j=N} X(j\Delta)h(t_i - j\Delta)$$

Hence,

$$\begin{cases} Y(t_1) = \lim_{N\to\infty}\lim_{\Delta\to 0}\sum_{j=-N}^{j=N} X(j\Delta)h(t_1 - j\Delta) \\ Y(t_2) = \lim_{N\to\infty}\lim_{\Delta\to 0}\sum_{j=-N}^{j=N} X(j\Delta)h(t_2 - j\Delta) \\ \quad\vdots \\ Y(t_n) = \lim_{N\to\infty}\lim_{\Delta\to 0}\sum_{j=-N}^{j=N} X(j\Delta)h(t_n - j\Delta) \end{cases}$$

Since $\{X(j\Delta)\}_{j=-N}^{N}$ is a Gaussian vector and random variables $(Y(t_1), Y(t_2), \ldots, Y(t_n))$ are linear combinations of random variables $\{X(j\Delta)\}_{j=-N}^{N}$, we conclude that they are also jointly Gaussian. ■

This theorem is a very important result and demonstrates one of the nice properties of Gaussian processes that makes them attractive. For a non-Gaussian process, knowledge of the statistical properties of the input process does not easily lead to the statistical properties of the output process. For Gaussian processes, we know that the output process of an LTI system will also be Gaussian. Hence, a complete statistical description of the output process requires only knowledge of the mean and autocorrelation functions of it. Therefore, it only remains to find the mean and the autocorrelation function of the output process and, as we have already seen in Section 4.2.4, this is an easy task.

Note that the above results hold for all Gaussian processes regardless of stationarity. Since a complete statistical description of Gaussian processes depends only on $m_X(t)$ and $R_X(t_1, t_2)$, we have also the following theorem.

Theorem 4.4.3. For Gaussian processes, WSS and strict stationarity are equivalent. ■

We also state the following theorem without proof. This theorem gives sufficient conditions for the ergodicity of zero-mean stationary Gaussian processes. For a proof, see Wong and Hajek (1985).

Theorem 4.4.4. A sufficient condition for the ergodicity of the stationary zero-mean Gaussian process $X(t)$ is that

$$\int_{-\infty}^{\infty} |R_X(\tau)|\,d\tau < \infty$$ ■

Parallel to the definition of jointly Gaussian random variables we can define *jointly Gaussian random processes.*

Definition 4.4.2. The random processes $X(t)$ and $Y(t)$ are *jointly Gaussian* if for all n, m and all $(t_1, t_2, \ldots, t_n)$ and $(\tau_1, \tau_2, \ldots, \tau_m)$, the random vector $(X(t_1), X(t_2), \ldots, X(t_n), Y(\tau_1)Y(\tau_2), \ldots, Y(\tau_m))$ is distributed according to an $n+m$ dimensional jointly Gaussian distribution.

We have also the following important theorem:

Theorem 4.4.5. For jointly Gaussian processes, uncorrelatedness and independence are equivalent.

Proof. This is also a straightforward consequence of the basic properties of Gaussian random variables as outlined in our discussion of jointly Gaussian random variables. ■

4.4.2 White Processes

The term *white process* is used to denote the processes in which all frequency components appear with equal power; i.e., the power-spectral density is a constant for all frequencies. This parallels the notion of "white light" in which all colors exist.

Definition 4.4.3. A process $X(t)$ is called a *white process* if it has a flat spectral density; i.e., if $\mathcal{S}_X(f)$ is a constant for all f.

The importance of white processes in practice stems from the fact that thermal noise can be closely modeled as a white process over a wide range of frequencies. Also, a wide range of processes used to describe a variety of information sources can be modeled as the output of LTI systems driven by a white process. Figure 4.19 shows the power spectrum of a white process.

If we find the power content of a white process using $\mathcal{S}_X(f) = C$, a constant, we will have

$$P_X = \int_{-\infty}^{\infty} \mathcal{S}_X(f)\, df = \int_{-\infty}^{\infty} C\, df = \infty$$

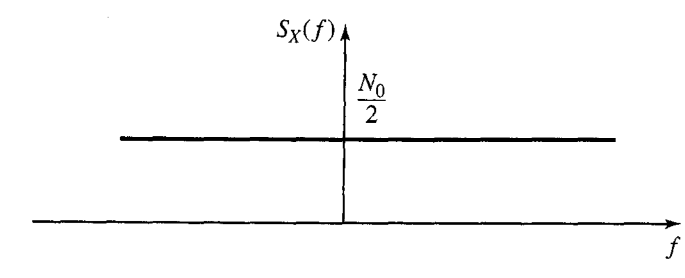

Figure 4.19 Power-spectral density of a white process.

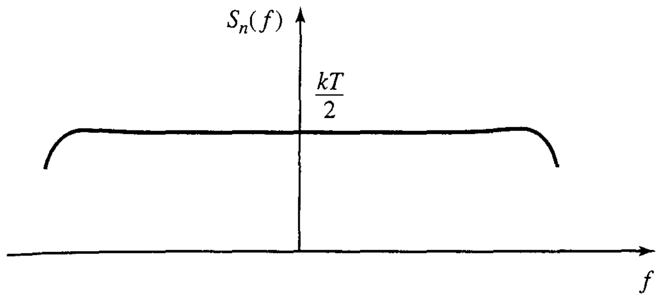

Figure 4.20 Power spectrum of thermal noise.

Obviously, no real physical process can have infinite power and, therefore, a white process is not a meaningful physical process. However, quantum mechanical analysis of the thermal noise shows that it has a power-spectral density given by

$$\mathcal{S}_n(f) = \frac{\hbar f}{2\left(e^{\frac{\hbar f}{kT}} - 1\right)} \tag{4.4.1}$$

in which $\hbar$ denotes *Planck's constant* (equal to 6.6×10^{-34} Joules $\times$ second) and k is *Boltzmann's constant* (equal to 1.38×10^{-23} Joules/Kelvin). T denotes the temperature in degrees Kelvin. This power spectrum is shown in Figure 4.20.

The above spectrum achieves its maximum at $f = 0$ and the value of this maximum is $\frac{kT}{2}$. The spectrum goes to zero as f goes to infinity, but the rate of convergence to zero is very slow. For instance, at room temperature ($T = 300$ K) $\mathcal{S}_n(f)$ drops to 90% of its maximum at about $f \approx 2 \times 10^{12}$ Hz, which is beyond the frequencies employed in conventional communication systems. From this we conclude that thermal noise, though not precisely white, for all practical purposes can be modeled as a white process with power spectrum equaling $\frac{kT}{2}$. The value kT is usually denoted by N_0 and, therefore, the power-spectral density of thermal noise is usually given as $\mathcal{S}_n(f) = \frac{N_0}{2}$, and sometimes referred to as the *two-sided power-spectral density*, emphasizing that this spectrum extends to both positive and negative frequencies. We will avoid this terminology throughout and simply use *power spectrum* or *power-spectral density*.

Looking at the autocorrelation function for a white process, we see that

$$R_n(\tau) = \mathcal{F}^{-1}\left[\frac{N_0}{2}\right] = \frac{N_0}{2}\delta(\tau)$$

This shows that for all $\tau \neq 0$ we have $R_X(\tau) = 0$; i.e., if we sample a white process at two points t_1 and t_2 ($t_1 \neq t_2$), the resulting random variables will be uncorrelated. If in addition to being white, the random process is also Gaussian, the sampled random variables will also be independent.

In short, the thermal noise that we will use in subsequent chapters is assumed to be a stationary, ergodic, zero-mean, white Gaussian process whose power spectrum is $\frac{N_0}{2} = \frac{kT}{2}$.

Noise-Equivalent Bandwidth. When white Gaussian noise passes through a filter, the output process, although still Gaussian, will not be white anymore. The filter characteristics determine the spectral properties of the output process, and we have

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = \frac{N_0}{2}|H(f)|^2$$

Now, if we want to find the power content of the output process, we have to integrate $\mathcal{S}_Y(f)$. Thus,

$$P_Y = \int_{-\infty}^{\infty} \mathcal{S}_Y(f)\,df = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\,df$$

Therefore, to determine the output power, we have to evaluate the integral $\int_{-\infty}^{\infty} |H(f)|^2\,df$. To do this calculation, we define B_{neq}, the *noise-equivalent bandwidth* of a filter with frequency response $H(f)$, as

$$B_{\text{neq}} = \frac{\int_{-\infty}^{\infty} |H(f)|^2\,df}{2\,H_{\max}^2} \tag{4.4.2}$$

where $H_{\max}$ denotes the maximum of $|H(f)|$ in the passband of the filter. Figure 4.21 shows $H_{\max}$ and B_{neq} for a typical filter.

Using the above definition, we have

$$\begin{aligned} P_Y &= \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\,df \\ &= \frac{N_0}{2} \times 2B_{\text{neq}}H_{\max}^2 \\ &= N_0\,B_{\text{neq}}H_{\max}^2 \end{aligned} \tag{4.4.3}$$

Therefore, by having B_{neq}, finding the output noise power becomes a simple task. The noise-equivalent bandwidth of filters and amplifiers is usually provided by the manufacturer.

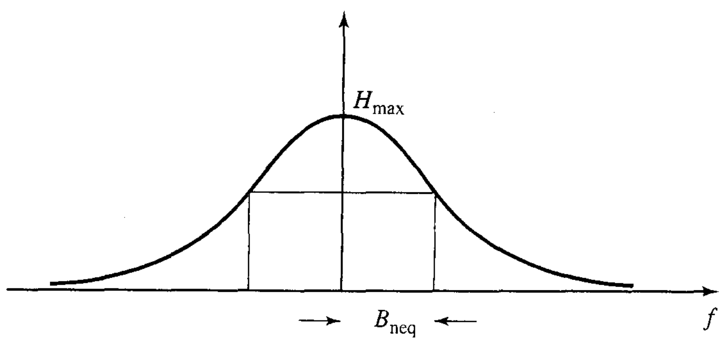

Figure 4.21 Noise-equivalent bandwidth of a typical filter.

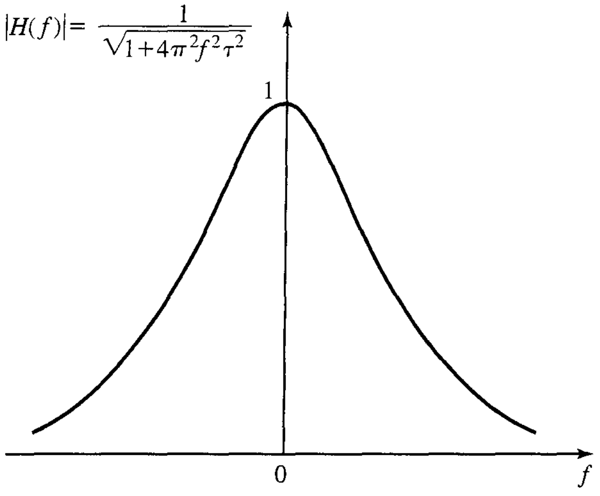

Figure 4.22 Frequency response of a lowpass RC filter.

Example 4.4.1

Find the noise-equivalent bandwidth of a lowpass RC filter.

Solution $H(f)$ for this filter is

$$H(f) = \frac{1}{1 + j2\pi fRC}$$

and is shown in Figure 4.22.

Defining $\tau = RC$, we have

$$|H(f)| = \frac{1}{\sqrt{1 + 4\pi^2 f^2 \tau^2}}$$

and, therefore, $H_{\max} = 1$. We also have

$$\begin{aligned}
\int_{-\infty}^{\infty} |H(f)|^2\, df &= 2\int_0^{\infty} \frac{1}{\sqrt{1 + 4\pi^2 f^2 \tau^2}}\, df \\
&\overset{u=2\pi f\tau}{=} 2\int_0^{\infty} \frac{1}{1+u^2} \times \frac{du}{2\pi\tau} \\
&= \frac{1}{\pi\tau} \times \frac{\pi}{2} \\
&= \frac{1}{2\tau}
\end{aligned}$$

Hence,

$$B_{\text{neq}} = \frac{\frac{1}{2\tau}}{2 \times 1} = \frac{1}{4RC}$$