Jensen's Inequality

Def: — A real-valued function f is <u>concave</u> on an interval I if

$$f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}$$

for all $x, y \in I$. It is <u>strictly</u> concave if $f\left(\frac{x+y}{2}\right) > \frac{f(x)+f(y)}{2}$

Thm — Suppose f is a continuous and strictly concave on I, and let $a_i > 0$, $1 \leq i \leq n$, such that $\sum_{i=1}^{n} a_i = 1$. Then

$$\sum_{i=1}^{n} a_i f(x_i) \leq f\left(\sum_{i=1}^{n} a_i x_i\right)$$

where $x_i \in I$. Equality occurs iff $x_1 = x_2 = \dots = x_n$

- Ex: Concave functions: $\log x$, $\sqrt{x}$
  Convex: x^2, e^x, $|x|$, $x \log x$

Proof by induction

$$a_1 f(x_1) + a_2 f(x_2) \leq f(a_1 x_1 + a_2 x_2)$$ — true by def. of concave function

$$\sum_{i=1}^{n} a_i f(x_i) = a_n f(x_n) + \sum_{i=1}^{n-1} a_i f(x_i)$$

$$= a_n f(x_n) + (1-a_n) \cdot \sum_{i=1}^{n-1} \frac{a_i}{1-a_n} f(x_i)$$

$$\leq a_n f(x_n) + (1-a_n) \cdot f\left(\sum_{i=1}^{n} \frac{a_i}{1-a_n} \cdot x_i\right)$$

$$\leq f\left(a_n x_n + (1-a_n) \sum_{i=1}^{n} \frac{a_i}{1-a_n} x_i\right)$$

$$= f\left(\sum_{i=1}^{n} a_i x_i\right)$$

Thm: $H(X,Y) \leq H(X) + H(Y)$ with equality iff X and Y are independent.

$$H(X) + H(Y) = -\sum_x p(x) \log p(x) - \sum_y p(y) \log p(y)$$

$$= -\sum_x \sum_y p(x,y) \log p(x) - \sum_y \sum_x p(x,y) \log p(y)$$

$$= -\sum_x \sum_y p(x,y) \log p(x) p(y)$$

$$H(X,Y) = -\sum_x \sum_y p(x,y) \log p(x,y)$$

$$H(X,Y) - H(X) - H(Y) = \sum_x \sum_y p(x,y) \log \frac{1}{p(x,y)} + \sum_x \sum_y p(x,y) \log p(x) p(y)$$

$$= \sum_x \sum_y p(x,y) \log \frac{p(x) p(y)}{p(x,y)} \qquad \log - \text{strictly convex}$$

$$\leq \log \left(\sum_x \sum_y \cancel{p(x,y)} \frac{p(x)p(y)}{\cancel{p(x,y)}} \right) \qquad - \text{Jensen's inequality}$$

$$= \log 1$$

equality for $\frac{p(x)p(y)}{p(x,y)} = c$