Random Processes

Def: A random process is a function $X: R \times \mathcal{J} \to R$, where $\mathcal{J}$ is a sample space.

Take the time instants $t_1, t_2, \dots, t_n$ $\quad n \geq 1$ and consider random variables $X(t_1, \cdot), X(t_2, \cdot), \dots, X(t_n, \cdot)$, denote these RV as $X_1 \quad X_2 \quad \dots \quad X_n$

We can then describe a random process by

$$f_{X_1, X_2 \dots X_n}(x_1, x_2, \dots, x_n) \quad \text{or} \quad f_{\bar{X}}(x_1, x_2, \dots, x_n)$$

Def: The autocorrelation function of X is given by

$$R(t_1, t_2) = E[X(t_1, \cdot), X(t_2, \cdot)] = \Big(\underbrace{E[X(t_1) \cdot X(t_2)] = E[X_1 X_2]}_{\text{or shorter}} \Big)$$

Def: The autocovariance function of X is

$$C(t_1, t_2) = E[(X(t_1) - \eta(t_1))(X(t_2) - \eta(t_2))]$$

$$\eta(t_i) = E[X(t_i)] \text{ - the mean at } t_1$$

Note: $C(t_1, t_2) = R(t_1, t_2) - \eta_1(t_1)\,\eta(t_2)$

$C(t, t) = \sigma^2(t)$ - the variance of the RV at t

$R(t, t) = E[X^2(t)]$ - average power - expected instantaneous power

Def: A process is first-order stationary if

$$f_X(x;t) = f_X(x;t+\tau) \quad \forall \tau$$

Note For stationary processes $\eta_X(t) = \text{const}$ $\sigma_X^2(t) = \text{const}$.

Def: A process X is n^{th}-order stationary if

$$f_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = f_X(x_1, x_2, \dots, x_n; t_1+\tau, t_2+\tau, \dots t_n+\tau) \quad \underline{\forall \tau}$$

Def: $X(t)$ is strict-sence stationary if it is stationary for $\forall n$

<u>SSS</u> - strict sense stationa

Def $X(t)$ is wide-sense stationary if

<u>WSS</u> - wide-sense stationary

a) $E[X(t)] = \eta_X(t) = \text{const}$

b) $E[R(t, t+\tau)] = R(\tau)$ time invariant autocorrelatio

Def: For $X(t)$ WSS, the power spectral density of X is

$S_X: \mathbb{R} \to \mathbb{C}$ where

$$S_X(\omega) = \int_{-\infty}^{+\infty} R_X(\tau) e^{-j\omega\tau} d\tau \qquad S_X(\omega) = \mathcal{F}\{R_X(\tau)\}$$

Note for discrete-time random processes with regular time intervals

$$S_X(\omega) = \sum_{k=-\infty}^{+\infty} R_X(kT) e^{-j\omega kT} \qquad R_X(k) = R_X(kT)$$

$S_X(\omega)$ is periodic with period 2π

Properties of power spectral density

a) $R_X(\tau) = \mathcal{F}^{-1}\{S_X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_X(\omega) e^{j\omega\tau} d\omega$

for discrete-time

$$R_X(kT) = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} S_X(\omega) e^{j\omega kT} d\omega$$

b) $S_X(\omega)$ is real and even

c) $S_X(\omega) \geq 0$ (because $R_X(\tau)$ is non-negative definite)

d $R_X(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_X(\omega) d\omega$ – average power

Ergodicity:

Def: Let $X(t)$ be WSS and let

$$\hat{\eta}_T = \frac{1}{2T}\int_{-T}^{T} X(t) dt$$

$$E[\hat{\eta}_T] = \frac{1}{2T}\int_{-T}^{T} E[X(t)] dt = \frac{1}{2T}\int_{-T}^{T} \eta\, dt = \eta$$

If $Var[\hat{\eta}_T] = E[(\hat{\eta}_T - \eta)^2] \to 0$ as $T \to \infty$

then $\hat{\eta}_T \xrightarrow{m.s.} \eta$ and we say that $X(t)$ is <u>mean-ergodic</u>

Ergodicity is a desirable property since expectations can be replaced by time averages

Def: Let $X(t)$ be WSS and $\hat{R}_T(\tau) = \frac{1}{2T}\int_{-T}^{T} X(t+\tau)X(t)\, dt$ $T \gg \tau$

If $\hat{R}_T(\tau) \xrightarrow{m.s.} R(\tau)$ we say that $X(t)$ is

<u>autocorrelation ergodic</u>

Gaussian and White Processes

- Thermal noise in electronic devices is modeled as white Gaussian process.

Def: A random process $X(t)$ is Gaussian if for all n and all $(t_1, t_2, \dots, t_n)$, the random variables $X(t_i)$ $i = 1, \dots, n$ have a jointly Gaussian density function

Remember

$$f_X(\underline{x}, \underline{t}) = \frac{1}{\sqrt{(2\pi)^2 \det(C(\underline{t}))}} e^{-\frac{1}{2}\left[(\underline{x} - \underline{\eta}(\underline{t}))C^{-1}(\underline{t})(\underline{x} - \underline{\eta}(\underline{t}))\right]}$$

$$\underline{x} = (x_1, x_2, \dots, x_n) \quad \underline{t} = (t_1, t_2, \dots, t_n)$$

$$C(\underline{t}) = \begin{bmatrix} C(t_1, t_1) & \dots & C(t_1, t_n) \\ & & \\ C(t_n, t_1) & \dots & C(t_n, t_n) \end{bmatrix}$$

Thm: For Gaussian process: WSS $\Leftrightarrow$ SSS, thus knowledge of the mean and autocorrelation function gives complete statistical description of the process.

Thm: If the Gaussian process $X(t)$ is passed through an linear, time invariant system (LTI), the output process $Y(t)$ will be also a Gaussian process.

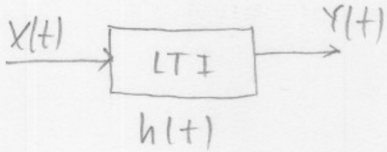
Proof: To prove that $Y(t)$ is Gaussian, we have to prove that for all n an all $(t_1, t_2, \dots, t_n)$ the vector $(Y(t_1), \dots Y(t_n))$ is Gaussian vector

Since $Y(t_i) = \int_{-\infty}^{+\infty} X(\tau) h(t_i - \tau) d\tau = \lim_{N \to \infty} \lim_{\Delta \to 0} \sum_{j=-N}^{j=N} X(j\Delta) h(t_i - j\Delta)$

the linear combination of $X(j\Delta)$ that are Gaussian is also gaussian.

Thm: A sufficient condition for the ergodicity of the stationary zero-mean Gaussian process $X(t)$ is $\int_{-\infty}^{+\infty} |R_X(\tau)| d\tau < \infty$

Random processes at the output of LTI systems



Def : A system is memoryless if $Y(t)$ depends only on $X(t)$

Thm : $\eta_Y(t) = \eta_X(t) * h(t)$

Thm : $R_Y(t_1, t_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R_X(t_1-\lambda_1, t_2-\lambda_2)\, h(\lambda_1)\, h(\lambda_2)\, d\lambda_1\, d\lambda_2$

Thm. $S_Y(\omega) = S_X(\omega)\, |H(\omega)|^2$ where $H(\omega) = \mathcal{F}\{h(\tau)\}$

Pf:

Sampling

Thm: Let $X(t)$ be a stationary (WSS) bandlimited process. and $S_X(f) \equiv 0$ for $|f| \geq W$, then

$$E\left[\left|X(t) - \sum_{k=-\infty}^{+\infty} X(kT_s) \frac{\sin \pi 2W(t-kT_s)}{\pi 2W(t-kT_s)}\right|^2\right] = 0$$

$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$

$$X(t) \overset{m.s}{=} \sum_{k=-\infty}^{+\infty} X(kT_s)\, \mathrm{sinc}(2W(t-kT_s))$$

Given two random processes $X(t)$ and $Y(t)$ we can define

$$R_{X,Y}(t_1,t_2) = E[X(t_1)\,Y(t_2)] \qquad \text{- cross-correlation}$$

$$C_{X,Y}(t_1,t_2) = E[(X(t_1)-\eta_X(t_1))(Y(t_2)-\eta_Y(t_2))] \qquad \text{- cross-covariance}$$

Def: The two processes $X(t)$ and $Y(t)$ are <u>orthogonal</u> if $R_{X,Y}(t_1,t_2)=0 \quad \forall t_1,t_2$

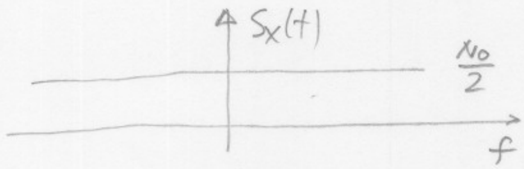
Def: Two processes are <u>uncorrelated</u> if $C_{X,Y}(t_1,t_2)=0 \quad \forall t_1,t_2$

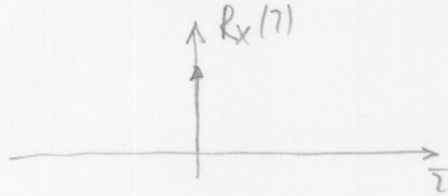
Thm: Two random processes are jointly Gaussian if for all n and m and all $(t_1,t_2,\dots t_n)$, $(\tau_1,\dots,\tau_m)$ the random vector

$$(X(t_1),X(t_2),\dots X(t_n),Y(\tau_1),\dots,Y(\tau_m)) \text{ is jointly Gaussian.}$$

Def: A process $X(t)$ is white if $S_X(\omega)=\text{const} \quad \forall \omega$.

Note: The power of $X(t)$ $\quad R_X(0) = \int_{-\infty}^{+\infty} S_X(f)\,df = \infty$





$$R_X(\tau) = \frac{N_0}{2}\delta(\tau)$$

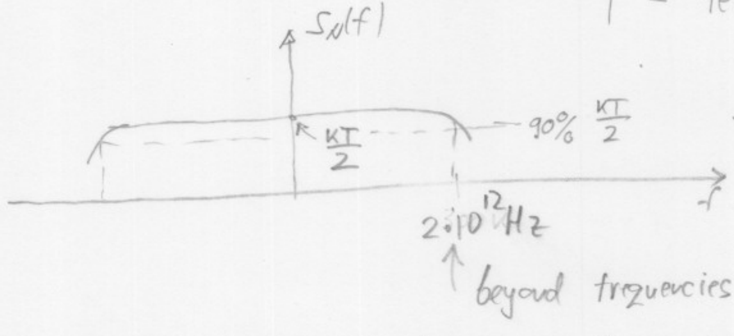
The origin of $\frac{N_0}{2}$:

- Quantum mechanics describes thermal noise as one with power spectral density $S_N(f) = \dfrac{\hbar f}{2\left(e^{\frac{\hbar f}{kT}} - 1\right)}$

$\hbar$ – Planck constant $\hbar = 6.6 \cdot 10^{-34}\ \frac{\text{Joules}}{\text{sec.}}$

k – Boltzman constant $k = 1.38 \cdot 10^{-23}\ \frac{\text{Joules}}{\text{Kelvin}}$

T – temperature in degrees Kelvin



for $k = 300K$

- Thus thermal noise is white $\quad \frac{N_0}{2} = \frac{kT}{2} \qquad N_0 = kT$