Solutions for Homework 5

1)

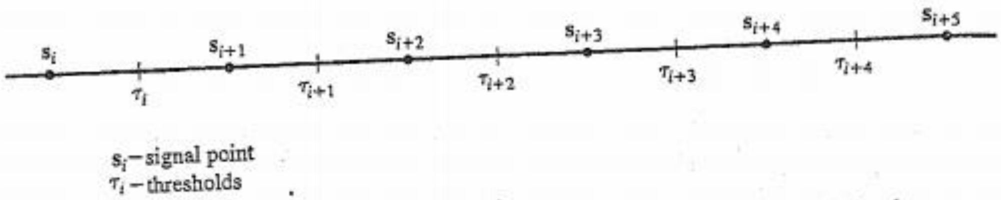

Figure 7.54 Placement of thresholds at midpoints of successive amplitude levels.

The placing of the thresholds as shown in Figure 7.54, helps in evaluating the probability of error. We note that if the mth amplitude level is transmitted, the demodulator output is

$$r = s_m + n = \sqrt{\mathcal{E}_g} A_m + n$$

where the noise variable n has zero mean and variance $\sigma_n^2 = N_0/2$. On the basis that all amplitude levels are equally likely a priori, the average probability of a symbol error is simply the probability that the noise variable n exceeds in magnitude one-half of the distance between levels. However, when either one of the two outside levels $\pm(M-1)$ is transmitted, an error can occur in one direction only. Thus, we have

$$\begin{aligned}
P_M &= \frac{M-1}{M} P\left(|r - s_m| > \sqrt{\mathcal{E}_g}\right) \\
&= \frac{M-1}{M} \frac{2}{\sqrt{\pi N_0}} \int_{\sqrt{\mathcal{E}_g}}^{\infty} e^{-x^2/N_0}\, dx \\
&= \frac{M-1}{M} \frac{2}{\sqrt{2\pi}} \int_{\sqrt{2\mathcal{E}_g/N_0}}^{\infty} e^{-x^2/2}\, dx \\
&= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{2\mathcal{E}_g}{N_0}}\right)
\end{aligned}$$

The error probability in Equation (7.6.30) can also be expressed in terms of the average transmitted power. From Equation (7.6.25), we note that

$$\mathcal{E}_g = \frac{3}{M^2-1} P_{av} T$$

By substituting for $\mathcal{E}_g$ in Equation (7.6.30), we obtain the average probability of a symbol error for PAM in terms of the average power as

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6 P_{av} T}{(M^2-1) N_0}}\right)$$

or, equivalently,

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\mathcal{E}_{av}}{(M^2-1)N_0}}\right)$$

where $\mathcal{E}_{av} = P_{av}T$ is the average energy.

In plotting the probability of a symbol error for M-ary signals such as M-ary PAM, it is customary to use the average SNR/bit as the basic parameter. Since $T = kT_b$ and $k = \log_2 M$, Equation (7.6.33) may be expressed as

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6(\log_2 M)\mathcal{E}_{bav}}{(M^2-1)N_0}}\right)$$

where $\mathcal{E}_{bav} = P_{av}T_b$ is the average bit energy and $\mathcal{E}_{bav}/N_0$ is the average SNR/bit. Figure 7.55 illustrates the probability of a symbol error as a function of $10\log_{10}\mathcal{E}_{bav}/N_0$ with M as a parameter. Note that the case $M=2$ corresponds to the error probability for binary antipodal signals. We also observe that the SNR/bit increases by over 4 dB for every factor of two increase in M. For large M, the additional SNR/bit required to increase M by a factor of two approaches 6 dB.

2)

Solution The received signal vector (one dimensional) for binary PAM is

$$r = \pm\sqrt{\mathcal{E}_b} + y_n(T)$$

where $y_n(T)$ is a zero-mean Gaussian random variable with variance $\sigma_n^2 = N_0/2$. Consequently, the conditional PDFs $f(r \mid s_m)$ for the two signals are

$$f(r \mid s_1) = \frac{1}{\sqrt{2\pi}\sigma_n} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma_n^2}$$

$$f(r \mid s_2) = \frac{1}{\sqrt{2\pi}\sigma_n} e^{-(r+\sqrt{\mathcal{E}_b})^2/2\sigma_n^2}$$

Then the metrics $\mathrm{PM}(\mathbf{r}, s_1)$ and $\mathrm{PM}(\mathbf{r}, s_2)$ defined by Equation (7.5.45) are

$$\begin{aligned}\mathrm{PM}(r, s_1) &= pf(r \mid s_1) \\ &= \frac{p}{\sqrt{2\pi}\sigma_n} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma_n^2} \\ \mathrm{PM}(\mathbf{r}, s_2) &= \frac{1-p}{\sqrt{2\pi}\sigma_n} e^{-(r+\sqrt{\mathcal{E}_b})^2/2\sigma_n^2}\end{aligned}$$

If $\mathrm{PM}(\mathbf{r}, \mathbf{s}_1) > \mathrm{PM}(\mathbf{r}, s_2)$, we select s_1 as the transmitted signal; otherwise, we select s_2. This decision rule may be expressed as

$$\frac{\mathrm{PM}(\mathbf{r}, \mathbf{s}_1)}{\mathrm{PM}(\mathbf{r}, \mathbf{s}_2)} \underset{s_2}{\overset{s_1}{\gtrless}} 1$$

But

$$\frac{\mathrm{PM}(\mathbf{r}, \mathbf{s}_1)}{\mathrm{PM}(\mathbf{r}, \mathbf{s}_2)} = \frac{p}{1-p} e^{[(r+\sqrt{\mathcal{E}_b})^2-(r-\sqrt{\mathcal{E}_b})^2]/2\sigma_n^2}$$

so that Equation (7.5.51) may be expressed as

$$\frac{(r+\sqrt{\mathcal{E}_b})^2-(r-\sqrt{\mathcal{E}_b})^2}{2\sigma_n^2} \underset{s_2}{\overset{s_1}{\gtrless}} \ln\frac{1-p}{p}$$

or, equivalently,

$$\sqrt{\mathcal{E}_b}r \underset{s_2}{\overset{s_1}{\gtrless}} \frac{\sigma_n^2}{2}\ln\frac{1-p}{p} = \frac{N_0}{4}\ln\frac{1-p}{p}$$

This is the final form for the optimum detector. It computes the correlation metric $C(\mathbf{r}, \mathbf{s}_1) = r\sqrt{\mathcal{E}_b}$ and compares it with the threshold $(N_0/4)\ln(1-p)/p$.

It is interesting to note that in the case of unequal prior probabilities, it is necessary to know not only the values of the prior probabilities but also the value of the power-spectral density N_0, in order to compute the threshold. When $p = 1/2$, the threshold is zero, and knowledge of N_0 is not required by the detector.

3)

a) A (5, 2) code is defined by

$$C = \{00000, 10100, 01111, 11011\}$$

It is very easy to verify that this code is linear. If the mapping between the information sequences and code words is given by

$$00 \rightarrow 00000$$
$$01 \rightarrow 01111$$
$$10 \rightarrow 10100$$
$$11 \rightarrow 11011$$

b)

Solution We have to find the code words corresponding to information sequences (10) and (01). These are (10100) and (01111), respectively. Therefore,

$$\mathbf{G} = \begin{bmatrix} 10100 \\ 01111 \end{bmatrix}$$

It is seen that for the information sequence (x_1, x_2), the code word is given by

$$(c_1, c_2, c_3, c_4, c_5) = (x_1, x_2)\mathbf{G}$$

or

$$c_1 = x_1$$
$$c_2 = x_2$$
$$c_3 = x_1 \oplus x_2$$
$$c_4 = x_2$$
$$c_5 = x_2$$

c) The codewords of the linear code of Example 9.5.1 are

$$\begin{aligned} \mathbf{c}_1 &= [\,0 \ 0 \ 0 \ 0 \ 0\,] \\ \mathbf{c}_2 &= [\,1 \ 0 \ 1 \ 0 \ 0\,] \\ \mathbf{c}_3 &= [\,0 \ 1 \ 1 \ 1 \ 1\,] \\ \mathbf{c}_4 &= [\,1 \ 1 \ 0 \ 1 \ 1\,] \end{aligned}$$

Since the code is linear the minimum distance of the code is equal to the minimum weight of the codewords. Thus,

$$d_{\min} = w_{\min} = 2$$

There is only one codeword with weight equal to 2 and this is $\mathbf{c}_2$.

d)

Solution Here

$$\mathbf{G} = \begin{bmatrix} 10100 \\ 01111 \end{bmatrix}$$

$$\mathbf{I} = \begin{bmatrix} 10 \\ 01 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 100 \\ 111 \end{bmatrix}$$

Noting that in the binary case $-\mathbf{P}' = \mathbf{P}'$, we conclude that

$$\mathbf{P}' = \begin{bmatrix} 11 \\ 01 \\ 01 \end{bmatrix}$$

and, therefore,

$$\mathbf{H} = \left[\begin{array}{c|c} 11 & 100 \\ 01 & 010 \\ 01 & 001 \end{array}\right]$$

4)

a) R = 4/7

b) The following table lists all the codewords of the (7,4) Hamming code along with their weight. Since the Hamming codes are linear $d_{\min} = w_{\min}$. As it is observed from the table the minimum weight is 3 and therefore $d_{\min} = 3$.

No.	Codewords	Weight
1	0000000	0
2	1000110	3
3	0100011	3
4	0010101	3
5	0001111	4
6	1100101	4
7	1010011	4
8	1001001	3
9	0110110	4
10	0101100	3
11	0011010	3
12	1110000	3
13	1101010	4
14	1011100	4
15	0111001	4
16	1111111	7

And as $d_{min} = 3$, the code can correct all error patterns with Hamming weight 1, or, the code corrects 1 error.

$$0 < t \leq \left[\frac{d-1}{2}\right] = 1$$

so, t=1, where t is the number of errors that the code can correct.