Solutions for Homework 4

1)

a. To show that the waveforms $f_n(t)$, $n = 1, \ldots, 3$ are orthogonal we have to prove that:

$$\int_{-\infty}^{\infty} f_m(t) f_n(t) dt = 0, \qquad m \neq n$$

Clearly:

$$\begin{aligned} c_{12} &= \int_{-\infty}^{\infty} f_1(t) f_2(t) dt = \int_0^4 f_1(t) f_2(t) dt \\ &= \int_0^2 f_1(t) f_2(t) dt + \int_2^4 f_1(t) f_2(t) dt \\ &= \frac{1}{4} \int_0^2 dt - \frac{1}{4} \int_2^4 dt = \frac{1}{4} \times 2 - \frac{1}{4} \times (4-2) \\ &= 0 \end{aligned}$$

Similarly:

$$\begin{aligned} c_{13} &= \int_{-\infty}^{\infty} f_1(t) f_3(t) dt = \int_0^4 f_1(t) f_3(t) dt \\ &= \frac{1}{4} \int_0^1 dt - \frac{1}{4} \int_1^2 dt - \frac{1}{4} \int_2^3 dt + \frac{1}{4} \int_3^4 dt \\ &= 0 \end{aligned}$$

and :

$$\begin{aligned} c_{23} &= \int_{-\infty}^{\infty} f_2(t) f_3(t) dt = \int_0^4 f_2(t) f_3(t) dt \\ &= \frac{1}{4} \int_0^1 dt - \frac{1}{4} \int_1^2 dt + \frac{1}{4} \int_2^3 dt - \frac{1}{4} \int_3^4 dt \\ &= 0 \end{aligned}$$

Thus, the signals $f_n(t)$ are orthogonal. It is also straightforward to prove that the signals have unit energy :

$$\int_{-\infty}^{\infty} |f_i(t)|^2 dt = 1, \;\; i = 1, 2, 3$$

Hence, they are orthonormal.

b. We first determine the weighting coefficients

$$x_n = \int_{-\infty}^{\infty} x(t) f_n(t) dt, \qquad n = 1, 2, 3$$

$$\begin{aligned} x_1 &= \int_0^4 x(t) f_1(t) dt = -\frac{1}{2} \int_0^1 dt + \frac{1}{2} \int_1^2 dt - \frac{1}{2} \int_2^3 dt + \frac{1}{2} \int_3^4 dt = 0 \\ x_2 &= \int_0^4 x(t) f_2(t) dt = \frac{1}{2} \int_0^4 x(t) dt = 0 \end{aligned}$$

$$x_3 = \int_0^4 x(t)f_3(t)dt = -\frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt + \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0$$

As it is observed, $x(t)$ is orthogonal to the signal wavaforms $f_n(t)$, $n = 1, 2, 3$ and thus it can not represented as a linear combination of these functions.

2)

A well-known result in estimation theory based on the minimum mean-squared-error criterion states that the minimum of $\mathcal{E}_e$ is obtained when the error is orthogonal to each of the functions in the series expansion. Hence :

$$\int_{-\infty}^{\infty}\left[s(t) - \sum_{k=1}^{K} s_k f_k(t)\right] f_n^*(t)dt = 0, \qquad n = 1, 2, ..., K \qquad (1)$$

since the functions $\{f_n(t)\}$ are orthonormal, only the term with $k = n$ will remain in the sum, so :

$$\int_{-\infty}^{\infty} s(t)f_n^*(t)dt - s_n = 0, \qquad n = 1, 2, ..., K$$

or:

$$s_n = \int_{-\infty}^{\infty} s(t)f_n^*(t)dt \qquad n = 1, 2, ..., K$$

The corresponding residual error $\mathcal{E}_e$ is :

$$\begin{aligned}
\mathcal{E}_{\min} &= \int_{-\infty}^{\infty}\left[s(t) - \sum_{k=1}^{K} s_k f_k(t)\right]\left[s(t) - \sum_{n=1}^{K} s_n f_n(t)\right]^* dt \\
&= \int_{-\infty}^{\infty}|s(t)|^2 dt - \int_{-\infty}^{\infty}\sum_{k=1}^{K} s_k f_k(t)s^*(t)dt - \sum_{n=1}^{K} s_n^* \int_{-\infty}^{\infty}\left[s(t) - \sum_{k=1}^{K} s_k f_k(t)\right] f_n^*(t)dt \\
&= \int_{-\infty}^{\infty}|s(t)|^2 dt - \int_{-\infty}^{\infty}\sum_{k=1}^{K} s_k f_k(t)s^*(t)dt \\
&= \mathcal{E}_s - \sum_{k=1}^{K}|s_k|^2
\end{aligned}$$

where we have exploited relationship (1) to go from the second to the third step in the above calculation.

3)

The amplitudes A_m take the values

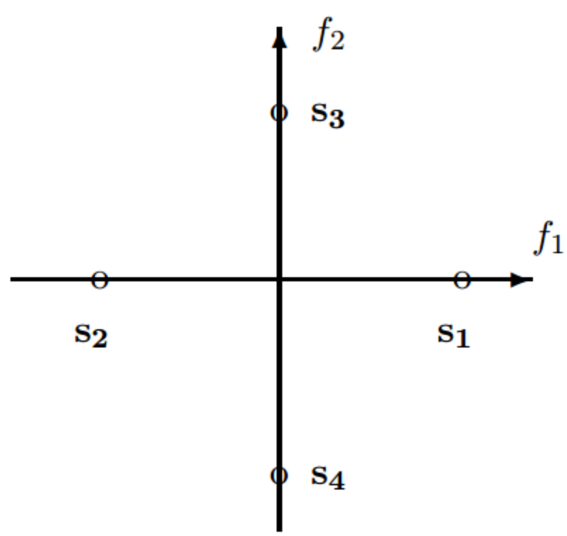
$$A_m = (2m - 1 - M)\frac{d}{2}, \qquad m = 1, \dots M$$

Hence, the average energy is

$$\begin{aligned}
\mathcal{E}_{av} &= \frac{1}{M}\sum_{m=1}^{M} s_m^2 = \frac{d^2}{4M}\mathcal{E}_g \sum_{m=1}^{M}(2m-1-M)^2 \\
&= \frac{d^2}{4M}\mathcal{E}_g \sum_{m=1}^{M}[4m^2 + (M+1)^2 - 4m(M+1)] \\
&= \frac{d^2}{4M}\mathcal{E}_g \left(4\sum_{m=1}^{M} m^2 + M(M+1)^2 - 4(M+1)\sum_{m=1}^{M} m\right) \\
&= \frac{d^2}{4M}\mathcal{E}_g \left(4\frac{M(M+1)(2M+1)}{6} + M(M+1)^2 - 4(M+1)\frac{M(M+1)}{2}\right) \\
&= \frac{M^2-1}{3}\frac{d^2}{4}\mathcal{E}_g
\end{aligned}$$

4)

$$\begin{aligned}
\mathbf{s}_1 &= \left(\sqrt{\mathcal{E}}, 0\right) \\
\mathbf{s}_2 &= \left(-\sqrt{\mathcal{E}}, 0\right) \\
\mathbf{s}_3 &= \left(0, \sqrt{\mathcal{E}}\right) \\
\mathbf{s}_4 &= \left(0, -\sqrt{\mathcal{E}}\right)
\end{aligned}$$



As we see, this signal set is indeed equivalent to a 4-phase PSK signal.

5) (Cauchy-Schwart inequality)

Theorem. *Let* $(a_1, a_2, \ldots, a_n)$ *and* $(b_1, b_2, \ldots, b_n)$ *be two sequences of real numbers, then*

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2,$$

Proof 1. Expanding out the brackets and collecting together identical terms we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j - a_jb_i)^2 = \sum_{i=1}^{n} a_i^2 \sum_{j=1}^{n} b_j^2 + \sum_{i=1}^{n} b_i^2 \sum_{j=1}^{n} a_j^2 - 2\sum_{i=1}^{n} a_ib_i \sum_{j=1}^{n} b_ja_j$$

$$= 2\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - 2\left(\sum_{i=1}^{n} a_ib_i\right)^2.$$

Because the left-hand side of the equation is a sum of the squares of real numbers it is greater than or equal to zero, thus

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_ib_i\right)^2.$$

Proof 2. Consider the following quadratic polynomial

$$f(x) = \left(\sum_{i=1}^{n} a_i^2\right)x^2 - 2\left(\sum_{i=1}^{n} a_ib_i\right)x + \sum_{i=1}^{n} b_i^2 = \sum_{i=1}^{n}(a_ix - b_i)^2.$$

Since $f(x) \geq 0$ for any $x \in \mathbb{R}$, it follows that the discriminant of $f(x)$ is negative, i.e.,

$$\left(\sum_{i=1}^{n} a_ib_i\right)^2 - \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \leq 0.$$