Solutions HW 3:

1) The capacity of a band limited Gaussian channel with noise variance N_0.

$$p(y_i|x_i) = p(n_i) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(y_i - x_i)^2}{2\sigma^2}}$$

$$I(X_N; Y_N) = H(Y_N) - H(Y_N|X_N)$$
$$= H(Y_N) - H(N_N)$$

x_i - Gaussian $\Rightarrow$ $I(X_N, Y_N)$ is maximized

$$H(N_N) = \sum_{i=1}^{N} H(N_i) = N \cdot \frac{1}{2} \log 2\pi e \sigma_n^2$$
$$H(Y_N) = \sum_{i=1}^{N} H(Y_i) = N \frac{1}{2} \log 2\pi e (\sigma_x^2 + \sigma_n^2)$$

$$I^*(X_N; Y_N) = N \cdot \frac{1}{2} \log\left(1 + \frac{\sigma_x^2}{\sigma_n^2}\right)$$

$$\sigma_n^2 = \frac{N_0}{2}$$

$$\sigma_x^2 = E(X_i^2) \qquad S = \frac{1}{T}\int_0^T E(x^2(t))\,dt = \frac{1}{T}\sum_{i=1}^{N} E(X_i^2)$$
$$S = \frac{1}{T} \cdot N \cdot \sigma_x^2$$

$$I^*(X_N; Y_N) = T \cdot W \log\left(1 + \frac{ST}{\frac{N}{2} \cdot N_0}\right)$$
$$= TW \log\left(1 + \frac{ST}{TW N_0}\right)$$
$$= TW \log\left(1 + \frac{S}{N_0 W}\right)$$

$$C = \lim_{T\to\infty} \frac{1}{T} I^*(X_N; Y_N) = W \log\left(1 + \frac{S}{N_0 W}\right)$$

$$S = \mathcal{E}_b \cdot C$$

2) (a), (b), and (d) ($C = W \log(1 + \frac{P}{N_0 W})$ whre $P = C\ E_b$.

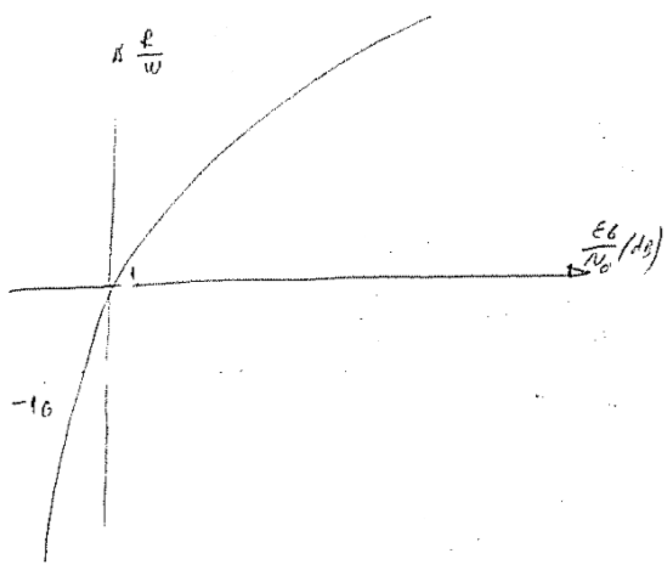
When either P/No or W tend to zero the capacity of the channel also tends to zero. When P/No or W tends to infinity, the capacity behaves differently:

when P/No tends to infinity, the capacity tends to infinity when W tends to infinity, the capacity goes to a limit determined by P/No.

(b) When W tends to infinity, the capacity goes to a limit determined by P/No.

$$\lim_{W \to \infty} W \log_2 \left(1 + \frac{P}{N_0 W}\right) = \frac{P}{N_0 \ln 2} = 1.4427 \frac{P}{N_0}$$

(d) When P/No tends to infinity, the capacity tends to infinity.



(c)

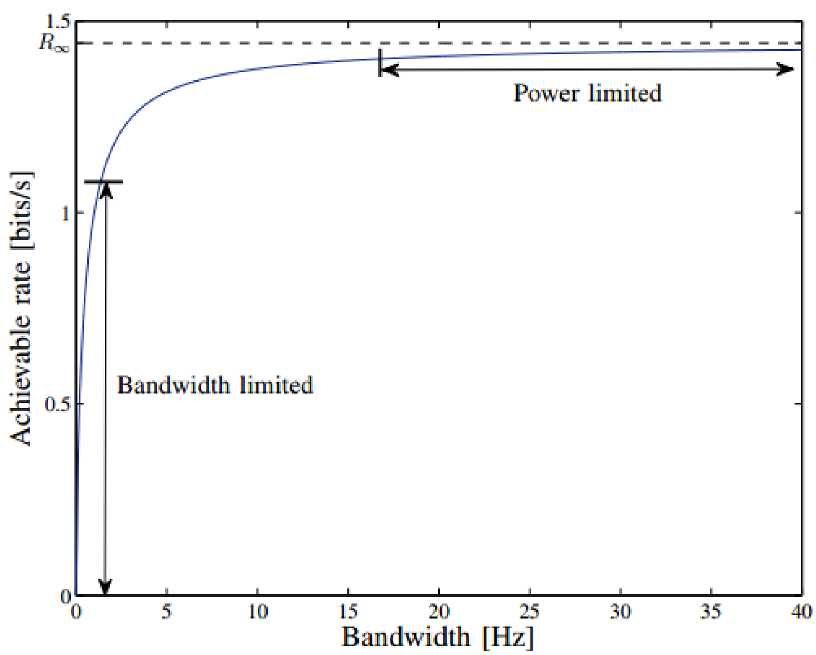
$$\frac{C}{W} \leq log_2\left(1+\frac{P}{N_0W}\right) = log_2\left(1+\frac{C}{W}\frac{E_b}{N_0}\right)$$

where $P = CE_b$. Using the above inequality, we can write

$$\frac{E_b}{N_0} \geq \frac{2^{\frac{C}{W}}-1}{\frac{C}{W}}.$$

(d)

illustrates the transmission rate as a function of the bandwidth for the case with $P/N_0 = 1$.



3)

Problem 6.39

Let X be a zero-mean Gaussian random variable with variance σ^2 and Y another zero-mean random variable such that

$$\int_{-\infty}^{\infty} y^2 f_Y(y)dy = \sigma^2$$

Applying the inequality $\ln z \leq z - 1$ to the function $z = \frac{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}}{f_Y(x)}$, we obtain

$$\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}\right) - \ln f_Y(x) \leq \frac{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}}{f_Y(x)} - 1$$

Multiplying the inequality by $f_Y(x)$ and integrating, we obtain

$$\int_{-\infty}^{\infty} f_Y(x)\left[\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{x^2}{2\sigma^2}\right]dx + h(Y) \leq 1 - 1 = 0$$

Hence,

$$\begin{aligned}
h(Y) &\leq -\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) + \frac{1}{2\sigma^2}\int_{-\infty}^{\infty} x^2 f_X(x)dx \\
&= \ln(\sqrt{2\pi\sigma^2}) + \frac{1}{2\sigma^2}\sigma^2 = \ln(e^{\frac{1}{2}}) + \ln(\sqrt{2\pi\sigma^2}) \\
&= h(X)
\end{aligned}$$