Problem 1: Two binary random variables X and Y are distributed according to the joint distribution $p(X = Y = 0) = p(X = 0, Y = 1) = p(X = Y = 1) = \frac{1}{3}$. Compute $H(X)$, $H(Y)$, $H(Y|X)$, $H(X|Y)$, and $H(X,Y)$.

Solution: First, we compute

$$p(X = 0) = p(X = 0, Y = 0) + p(X = 0, Y = 1) = \frac{2}{3} \quad \text{(mutually exclusive, marginal)}$$

and also

$$p(X = 1) = p(X = 1, Y = 0) + p(X = 1, Y = 1) = \frac{1}{3} \quad \text{(mutually exclusive, marginal)}$$

Then, we compute $H(X)$ as

$$\begin{aligned} H(X) &= \sum_{x_i \in X} p(x_i) \log_2 \frac{1}{p(x_i)} \\ &= p(X=0) \log_2 \frac{1}{p(X=0)} + p(X=1) \log_2 \frac{1}{p(X=1)} \\ &= \frac{2}{3} \log_2 \frac{3}{2} + \frac{1}{3} \log_2 3 \quad (\text{or } \log_2 3 - \frac{2}{3} \log_2 2) \\ &\approx \frac{2}{3} \cdot 0.585 + \frac{1}{3} \cdot 1.585 \\ &\approx 0.9183 \quad \text{bits} \end{aligned}$$

Similarly, we can also compute

$$p(Y = 0) = p(X = 0, Y = 0) + p(X = 1, Y = 0) = \frac{1}{3}$$

and

$$p(Y = 1) = p(X = 0, Y = 1) = p(X = 1, Y = 1) = \frac{2}{3}$$

Similarly, we can compute $H(Y)$ as

$$\begin{aligned} H(Y) &= \sum_{y_i \in Y} p(y_i) \log_2 \frac{1}{p(y_i)} \\ &= p(Y=0) \log_2 \frac{1}{p(Y=0)} + p(Y=1) \log_2 \frac{1}{p(Y=1)} \\ &= \frac{1}{3} \log_2 3 + \frac{2}{3} \log_2 \frac{3}{2} \\ &\approx 0.9183 \quad \text{bits} \end{aligned}$$

Since we know that $p(x_i|y_i) = \frac{p(x_i, y_i)}{p(y_i)}$, we can compute $H(X|Y)$ as

$$\begin{aligned} H(X|Y) &= \sum_{x_i, y_i \in X, Y} p(x_i, y_i) \log_2 \frac{1}{p(x_i|y_i)} \\ &= \sum_{x_i, y_i \in X, Y} p(x_i, y_i) \log_2 \frac{p(y_i)}{p(x_i, y_i)} \\ &= \frac{1}{3} \log_2 \frac{\frac{1}{3}}{\frac{1}{3}} + 0 + \frac{1}{3} \log_2 \frac{\frac{2}{3}}{\frac{1}{3}} + \frac{1}{3} \log_2 \frac{\frac{2}{3}}{\frac{1}{3}} \\ &= \frac{2}{3} \log_2 2 = \frac{2}{3} \approx 0.67 \quad \text{bits} \end{aligned}$$

Similarly, we know that $p(y_i|x_i) = \frac{p(x_i,y_i)}{px_i}$, we can calculate $H(Y|X)$ as

$$\begin{aligned} H(Y|X) &= \sum_{x_i,y_i \in X,Y} p(x_i,y_i)\log_2 \frac{1}{p(y_i|x_i)} \\ &= \sum_{x_i,y_i \in X,Y} p(x_i,y_i)\log_2 \frac{p(x_i)}{p(x_i,y_i)} \\ &= \frac{1}{3}\log_2 \frac{\frac{2}{3}}{\frac{1}{3}} + 0 + \frac{1}{3}\log_2 \frac{\frac{2}{3}}{\frac{1}{3}} + \frac{1}{3}\log_2 \frac{\frac{1}{3}}{\frac{1}{3}} \\ &= \frac{2}{3}\log_2 2 \approx 0.67 \quad \text{bits} \end{aligned}$$

Then we can compute mutual entropy $H(X,Y)$ as

$$H(X,Y) = H(X) + H(Y|X) = \log_2 3 - \frac{2}{3} + \frac{2}{3} \approx 1.585$$

Can also compute it as

$$H(X,Y) = \sum_{x_i,y_i \in X,Y} p(x_i,y_i)\log_2 \frac{1}{p(x_i,y_i)} \approx 1.585$$

And mutual information $I(X;Y)$ can be evaluated as

$$I(X;Y) = H(X) - H(X|Y) = 0.918 - 0.67 = 0.248$$

or compute it as

$$I(X;Y) = \sum_{x_i,y_i \in X,Y} p(x_i,y_i)\log_2 \frac{p(x_i|y_i)}{p(x_i)} = \sum_{x_i,y_i \in X,Y} p(x_i,y_i)\log_2 \frac{p(x_i,y_i)}{p(x_i)p(y_i)}$$

Problem 2: Let X and Y denote two jointly distributed discrete valued random variables. Show that $H(X) = -\sum_{x,y} p(x,y)\log_2 p(x)$ and $H(Y) = -\sum_{x,y} p(x,y)\log_2 p(y)$.

Solution: Since $p(x) = \sum_y p(x,y)$, we have

$$H(X) = -\sum_x p(x)\log_2 p(x) = -\sum_x \sum_y p(x,y)\log_2 p(x) = -\sum_{x,y} p(x,y)\log_2 p(x)$$

Similarly, since $p(y) = \sum_x p(x,y)$, we can also prove that

$$H(Y) = -\sum_y p(y)\log_2 p(y) = -\sum_y \sum_x p(x,y)\log_2 p(y) = -\sum_{x,y} p(x,y)\log_2 p(y)$$

Problem 3: Let X and Y denote two jointly distributed discrete valued random variables. Show that

$$H(X,Y) \leq H(X) + H(Y)$$

and when does the equality hold?

Solution: Assume we already know that $H(X,Y) = H(X) + H(Y|X)$, then we essentially need to prove $H(Y|X) \leq H(Y)$, which can be obtained as below

$$\begin{aligned}
H(Y|X) - H(Y) &= \sum_{x,y} p(x,y)\log_2 \frac{1}{p(y|x)} - \sum_{x,y} p(x,y)\log_2 \frac{1}{p(y)} \\
&= \sum_{x,y} p(x,y)\log_2 \frac{p(y)}{p(y|x)} \\
&= \sum_{x,y} p(x,y)\ln \frac{p(y)}{p(y|x)} / \ln 2 \\
&= \frac{1}{\ln 2}\sum_{x,y} p(x,y)\ln \frac{p(y)}{p(y|x)} \quad (\text{use } \ln\alpha \le \alpha - 1) \\
&\le \frac{1}{\ln 2}\sum_{x,y} p(x,y)(\frac{p(y)}{p(y|x)} - 1) \\
&= \frac{1}{\ln 2}\left(\sum_{x,y} p(x,y)\frac{p(y)p(x)}{p(x,y)} - \sum_{x,y} p(x,y)\right) \\
&= \frac{1}{\ln 2}\left(\sum_{y} p(y)\sum_{x} p(x) - 1\right) \\
&= \frac{1}{\ln 2}\left(\sum_{y} p(y)\cdot 1 - 1\right) \\
&= \frac{1}{\ln 2}(1-1) = 0
\end{aligned}$$

which implies $H(Y|X) \le H(Y)$ and $H(X,Y) \le H(X) + H(Y)$. The equality happens when $p(y) = p(y|x)$ (i.e., $p(x,y) = p(x)p(y)$, X and Y are independent).

Problem 4: Let X denote a binary random variable with $p(X=0) = 1 - p(X=1) = p$ and let Y be a binary random variable that depends on X through $p(Y=1|X=0) = p(Y=0|X=1) = \epsilon$.

(a) Find $H(X)$, $H(Y)$, $H(X|Y)$, $H(Y|X)$, $H(X,Y)$, and $I(X;Y)$.

(b) For a fixed p, which ϵ minimizes $I(X;Y)$?

Solution: We first compute

$$\begin{aligned}
H(X) &= \sum_{x} p(x)\log_2 \frac{1}{p(x)} \\
&= p(X=0)\log_2 \frac{1}{p(X=0)} + p(X=1)\log_2 \frac{1}{p(X=1)} \\
&= p\log_2 \frac{1}{p} + (1-p)\log_2 \frac{1}{(1-p)}
\end{aligned}$$

Then, we calculate

$$p(Y=0) = p(X=0)p(Y=0|X=0) + p(X=1)p(Y=0|X=1) = p(1-\epsilon) + (1-p)\epsilon = p + \epsilon - 2p\epsilon$$

and

$$p(Y=1) = p(X=0)p(Y=1|X=0) + p(X=1)p(Y=1|X=1) = p\epsilon + (1-p)(1-\epsilon) = 1 - p - \epsilon + 2p\epsilon$$

Next we have

$$\begin{aligned}
H(Y) &= p(Y=0)\log_2 \frac{1}{p(Y=0)} + p(Y=1)\log_2 \frac{1}{p(Y=1)} \\
&= (p + \epsilon - 2p\epsilon)\log_2 \frac{1}{(p+\epsilon-2p\epsilon)} + (1 - p - \epsilon + 2p\epsilon)\log_2 \frac{1}{(1-p-\epsilon+2p\epsilon)}
\end{aligned}$$

In order to compute $H(Y|X)$, we first compute $p(x,y) = p(x)p(y|x)$ as

$$p(X=0,Y=0) = p(X=0)p(Y=0|X=0) = p(1-\epsilon)$$

$$p(X=0,Y=1) = p(X=0)p(Y=1|X=0) = p\epsilon$$

$$p(X=1,Y=0) = p(X=1)p(Y=0|X=1) = (1-p)\epsilon$$

$$p(X=1,Y=1) = p(X=1)p(Y=1|X=1) = (1-p)(1-\epsilon)$$

Then we compute

$$\begin{aligned}
H(Y|X) &= \sum p(x,y)\log_2 \frac{1}{p(y|x)} \\
&= p(X=0,Y=0)\log_2 \frac{1}{p(Y=0|X=0)} + p(X=0,Y=1)\log_2 \frac{1}{p(Y=1|X=0)} \\
&\quad + p(X=1,Y=0)\log_2 \frac{1}{p(Y=0|X=1)} + p(X=1,Y=1)\log_2 \frac{1}{p(Y=1|X=1)} \\
&= p(1-\epsilon)\log_2 \frac{1}{1-\epsilon} + p\epsilon\log_2 \frac{1}{\epsilon} + (1-p)\epsilon\log_2 \frac{1}{\epsilon} + (1-p)(1-\epsilon)\log_2 \frac{1}{1-\epsilon} \\
&= (1-\epsilon)\log_2 \frac{1}{(1-\epsilon)} + \epsilon\log_2 \frac{1}{\epsilon}
\end{aligned}$$

The mutual entropy $H(X,Y)$ can be computed as

$$\begin{aligned}
H(X,Y) &= H(X) + H(Y|X) \\
&= p\log_2 \frac{1}{p} + (1-p)\log_2 \frac{1}{(1-p)} + (1-\epsilon)\log_2 \frac{1}{(1-\epsilon)} + \epsilon\log_2 \frac{1}{\epsilon}
\end{aligned}$$

Since we also know that $H(X,Y) = H(Y) + H(X|Y)$, we can also compute

$$\begin{aligned}
H(X|Y) &= H(X,Y) - H(Y) \\
&= p\log_2 \frac{1}{p} + (1-p)\log_2 \frac{1}{(1-p)} + (1-\epsilon)\log_2 \frac{1}{(1-\epsilon)} + \epsilon\log_2 \frac{1}{\epsilon} \\
&\quad -(p+\epsilon-2p\epsilon)\log_2 \frac{1}{(p+\epsilon-2p\epsilon)} + (1-p-\epsilon+2p\epsilon)\log_2 \frac{1}{(1-p-\epsilon+2p\epsilon)}
\end{aligned}$$

Finally, the mutual information $I(X;Y)$ can be computed as

$$\begin{aligned}
I(X;Y) &= H(X) - H(X|Y) = H(Y) - H(Y|X) \\
&= (p+\epsilon-2p\epsilon)\log_2 \frac{1}{(p+\epsilon-2p\epsilon)} + (1-p-\epsilon+2p\epsilon)\log_2 \frac{1}{(1-p-\epsilon+2p\epsilon)} \\
&\quad -(1-\epsilon)\log_2 \frac{1}{(1-\epsilon)} + \epsilon\log_2 \frac{1}{\epsilon}
\end{aligned}$$

Given a fixed p, $\epsilon = \frac{1}{2}$ minimize $I(X;Y)$ (by computing the derivative $dI(X;Y)/d\epsilon$).

Problem 5: Let X_1, X_2, . . . , X_n be drawn according to $p(x_1, x_2, ..., x_n)$. Show that

$$H(X_1, X_2, ..., X_n) \le H(X_1) + H(X_2) + \cdots + H(X_n)$$

Solution: Assume we know that

$$H(X) = \sum_{x \in X} p(x)\log_2 \frac{1}{p(x)}$$

and

$$p(x_1, x_2, ..., x_n) = p(x_1)p(x_1|x_2)\cdots p(x_n|x_1, ..., x_{n-1})$$

Then we have

$$\begin{aligned}
H(X_1, X_2, ..., X_n) &= \sum_{x_1,x_2,...,x_n \in X_1,X_2,...,X_n} p(x_1, x_2, ..., x_n) \log_2 \frac{1}{p(x_1, x_2, ..., x_n)} \\
&= \sum_{x_1,x_2,...,x_n \in X_1,X_2,...,X_n} p(x_1, x_2, ..., x_n) \log_2 \frac{1}{p(x_1)p(x_1|x_2)\cdots p(x_n|x_1, ..., x_{n-1})} \\
&= \sum_{x_1,x_2,...,x_n \in X_1,X_2,...,X_n} p(x_1, x_2, ..., x_n) \left(\log_2 \frac{1}{p(x_1)} + \log_2 \frac{1}{p(x_1|x_2)} + \cdots + \log_2 \frac{1}{p(x_n|x_1, ..., x_{n-1})} \right) \\
&= \sum p(x_1, x_2, ..., x_n) \log_2 \frac{1}{p(x_1)} + \cdots + \sum p(x_1, x_2, ..., x_n) \log_2 \frac{1}{p(x_n|x_1, ..., x_{n-1})} \\
&= H(X_1) + H(X_2|X_1) + \cdots + H(X_n|X_1, ..., X_{n-1})
\end{aligned}$$

In Problem 3, we already proved that $H(Y) \leq H(Y|X)$, it is easy to see that

$$\begin{aligned}
H(X_1, X_2, ..., X_n) &= H(X_1) + H(X_2|X_1) + \cdots + H(X_n|X_1, ..., X_{n-1}) \\
&\leq H(X_1) + H(X_2) + \cdots + H(X_n)
\end{aligned}$$

And the equality holds when X_1, X_2, ..., X_n are independent.