**Demo #3**

**Controller Design w/ Output Feedback**

\[
G(s) = \frac{K mw/M}{s^2 + \frac{B}{M} s + \frac{k}{M}}
\]

**Constraint:** Unlike in Lab #2, where both position and velocity signals were available for feedback, Lab #3 will only use the position signal to control the system. This constraint makes controlling the system more difficult.
Lab #3 will use

Two Different Controller Types

1) Phase-lead, \( G_c(\sigma) = \frac{K(\sigma + z)}{(\sigma + p)} \)

2) PID, \( G_c(\sigma) = K_p + \frac{K_i}{\sigma} + K_d \sigma \)

Pole/Zero Pattern of \( G(\sigma) \)

- X - j20
- j10
- j10
-2
- j20

X - j20
Phase-lead Controller Design

Design Specs: Speed up the system response to allow $T_s$ to be less than 0.5 seconds. (No constraint on ess!)

Allowable Region in $\rho$-plane:

$$T_s = \frac{\tau}{\sigma} < 0.5 \Rightarrow \frac{4}{0.5} = 8 < \sigma$$

:. Want the closed-loop poles to be to the left of the vertical line at $\rho = -\sigma = -8$.

The zero in the phase-lead controller is designed to attract the root locus pattern further into the left-half plane.
Possible Root Locus patterns with a phase-lead controller.

No matter which pattern, there will always be a 3rd pole on the real axis between the zero and the pole of the controller.
Suppose we locate the controller zero to the left of \( \sigma = -8 \) to help meet the settling time specification.

**Note:** The asymptote centroid formula should help us determine the phase-lead zero and pole locations.

(Let \( p = p^* = -\sigma_j + j\omega_d \).

\[
\sigma_A = \frac{(-\sigma_j - \sigma_k - p) - (-\varepsilon)}{3 - 1}
\]

We should select \( \varepsilon \) and \( p \) to force \( \sigma_A \) to the left of \( \sigma = -8 \), if possible.
Suppose our plant has poles at \( \rho = -2.5 \pm j17.5 \). \( \Rightarrow \sigma_1 = 2.5 \)

\[ \therefore \sigma_A = \frac{(-2.5 - 2.5 - \rho) - (-2)}{2} \]

Suppose we let the zero be at \( \rho = -10 \). \( \Rightarrow z = 10 \).

If we want \( \sigma_A \) to be near \(-15\), then we can determine a pole location that yields \( \sigma_A = -15 \).

\[ \sigma_A = \frac{(-2.5 - 2.5 - \rho) + 2}{2} = \frac{-5 - \rho + 10}{2} = -15 \]

\[ \Rightarrow -\rho + 5 = -30 \quad \therefore \rho = 35. \]

\[ G_c(\rho) = \frac{K(\rho + 10)}{\rho(\rho + 35)} \]
Let's find a gain value and see if the resulting complex poles are to the left of ρ = -8.

Suppose we let the real pole (the closed-loop pole between ρ = -P = -35 and ρ = -Z = -10) be at ρ = -17.5.

$$K = \frac{|\rho + 2.5 - j17.5||\rho + 2.5 + j17.5||\rho + 35|}{|\rho + 10||4000|}$$

(We have assumed a $K_{hu} = 4000$ and a total system mass of 1 kg.)

$$G(\rho) = \frac{4000}{(\rho + 2.5)^2 + (17.5)^2}$$

$$K = \frac{|-15 - j17.5||-15 + j17.5||17.5|}{|-7.5||4000|} \approx 0.3$$
Check the closed-loop poles for this value of $K$.

$$1 + K \frac{(\alpha + 10)}{(\alpha + 35)} \cdot \frac{4000}{[(\alpha + 2.5)^3 + (17.5)^3]} = 0$$

Numerator of the characteristic equations:

$$(\alpha + 35)(\alpha^2 + 5\alpha + 312.5) + (0.3)(\alpha + 10)(4000) = 0$$

$\alpha^3 + 40\alpha^2 + 1687.5\alpha + 22937.5 = 0$

Roots: $-17.7, -11.1 \pm j34.2$

5 of Complex Roots:

$$\theta = \tan^{-1}\left(\frac{34.2}{11.1}\right) = 72^\circ, s = \cos(72^\circ) = 0.31$$

With the zero at $\alpha = -10$, the percent overshoot will be much worse than a pure, 2-pole system with $s = 0.31$.

(See Figure 5.13a) The third pole at $\alpha = -17.7$ will help reduce the overshoot some.
The steady-state error, $e_{ss}$, for this phase-lead controlled system might be very large. 

\[ A = 2000 \quad \text{(Ref. Input)} \]

\[ |e_{ss}| \]

The error due to a unit step is 

\[ e[s] = \frac{1}{1 + G_c(s)G(s)} \cdot \frac{1}{s} \]

F.V.T. $\Rightarrow$ 

\[ e_{ss} = \lim_{s \to 0} e[s] = \lim_{s \to 0} \frac{1}{1 + G_cG} \]

\[ e_{ss} = \frac{1}{1 + \frac{0.2(10)}{(35)} \cdot \frac{4000}{[(2.5)^2 + (17.5)^2]}} = \frac{1}{1 + 1.1} = 0.48 \]

What is the approximate position of our mass, in the steady-state, if $A = 2000$ counts?

\[ e(t) = r(t) - x(t) \Rightarrow (0.48)(2000) = 2000 - x_{ss} \]

$\therefore$ 

\[ x_{ss} \approx 1040 \quad \text{Counts} \]
**PID Controller Design**

The integral term in the PID controller can be used to reduce or eliminate errors due to constant inputs.

**Design Specs:**
1) $T_s < 0.5$ seconds
2) $\%$ of complex poles $> 0.7$
   ($\theta$ of complex poles $< 45^\circ$)

$$G_c(s) = K_p + \frac{K_i}{s} + K_d s$$

$$= K_d s^2 + K_p s + K_i$$

$$G_c(s) = \frac{K_d (s^2 + \frac{K_p}{K_i} s + \frac{K_i}{K_d})}{s}$$

Ideal $G_c(s)$: 2 zeros & 1 pole ($\text{Pole@Origin}$)
Possible Root Locus Patterns w/ PID cont.
Suppose we locate the zeros in the PID controller at $s = -20$ and $s = -20$ to help simplify the design.

\[ x \begin{array}{c} j 20 \\ -20 \end{array} \]

\[ -20 \quad -13 \]

\[ x \quad j 20 \]

Suppose we find the gain value, $K_D$, in the PID controller to place a closed-loop pole at $\rho_D = -13$.

\[ K_D = \frac{|s + 2.5 - j 17.5|/|s + 2.5 + j 17.5|/10|}{|s + 20|^2 (4000)} \]

\[ \rho = -13 \]

\[ K_D = \frac{|-10.5 - j 17.5|/-10.5 + j 17.5|/13|}{|7|^2 (4000)} = 0.028 \]
Thus,

$$G_e[\omega] = \frac{0.028(\sigma + 2\sigma)}{\sigma}$$

$$G_e[\omega] = \frac{0.028(\sigma^2 + 400\sigma + 4000)}{\sigma}$$

$$G_e[\omega] = 0.028\sigma + 1.12 + 11.2\left(\frac{1}{\sigma}\right)$$

\[\therefore K_d = 0.028, \quad K_p = 1.12, \quad K_z = 11.2\]

Find the closed-loop poles for these PID parameter values.

$$1 + G_e[\omega] G[\omega] = 0$$

$$1 + \frac{0.028(\sigma^2 + 400\sigma + 4000)}{\sigma} \cdot \frac{4000}{(\sigma^2 + 5\sigma + 312.5)} = 0$$

**Numerator:**

$$\sigma(\sigma^2 + 5\sigma + 312.5) + 0.028(4000)(\sigma^2 + 900\sigma + 4000) = 0$$

$$\Rightarrow \sigma^3 + 117\sigma^2 + 4,792.2\sigma + 44,800 = 0$$
Roots: -13, -52.0 ± j27.1

A gain value of 0.028 may be a little high. (The complex roots, in the closed-loop system, have a pretty small angle, which is obviously less than 45°.)

You may need to adjust K₀ in your system, in order not to hit the system too hard.