Pole Placement & Time Response

\[ M \ddot{x} + B \dot{x} + kx = K_{hw} u \]

\[ \Rightarrow \ x'' = \frac{1}{M} (-B \dot{x} - kx + K_{hw} u) \]

**Block Diagram (Open Loop)**

Open-loop Transfer Function:

\[ T_{o}(s) = \frac{X(s)}{U(s)} = \frac{K_{hw}/M}{s^2 + \frac{B}{M} s + \frac{k}{M}} \]
Open-Loop Pole Locations:

\[ s = -\sigma + j\omega_d \]

How can we change the pole locations without changing B, K, or \( K_{hu} \)?

Electrically \( \Rightarrow \) Measure position, velocity, scale these measured signals, and feed them back into the input, \( u(t) \).
Block Diagram (Closed-loop):

Input Structure:
\[ u(t) = -k_i x(t) - k_e x'(t) + k_{pf} r(t) \]

Applied to System:
\[ Mx'' = -8x' - 2x + K_{hw} (-k_i x - k_e x' + k_{pf} r) \]
\[ Mx'' + Bx' + Kx = K_{hw} (-k_i x - k_e x' + k_{pf} r) \]
\[ Mx'' + (B + K_{hw} k_e)x' + (K + K_{hw} k_i)x = K_{hw} k_{pf} r \]

\( \bar{B} = \text{Modified damping coefficient (from velocity feedback)} \)
\( \bar{K} = \text{Modified spring constant (from position feedback)} \)
Closed-loop Transfer Function:

\[ T_c(\sigma) = \frac{X(\sigma)}{R(\sigma)} = \frac{(K_{nu}K_p/M)}{\sigma^2 + \left(\frac{B + K_{nu}K_i}{M}\right)\sigma + \left(\frac{K + K_{nu}K_i}{M}\right)} \]

Standard Denominators:

\[ d(\sigma) = (\sigma + \sigma_1)^2 + \omega_n^2 \]

\[ = \sigma^2 + 2\sigma_1\sigma + (\sigma_1^2 + \omega_n^2) \]

- OR -

\[ d(\sigma) = \sigma^2 + 2j\omega_n\sigma + \omega_n^2 \]

\[ \Rightarrow \quad \sigma = j\omega_n, \quad \omega_n^2 = \sigma^2 + \omega_n^2 \]

\[ \sigma \text{-plane} \]
Suppose we want the following design specifications:

1) Natural frequency, \( \omega_n \), of 35.

2) Percent overshoot, \( P.O. \), of 10%.

3) Small or zero steady-state error \( \rightarrow \) D.C. gain equal or near one.

Relating specifications to our physical system.

Spec 1) \[ \omega_n^2 = \frac{k + K_{hw} k_i}{M} = (35)^2 \]

\[ \therefore K_i = \frac{(\omega_n^2 M - k)}{K_{hw}} \]

Position Feedback Gain

Spec 2) \[ 10\% \text{ Overshoot} \Rightarrow s = 0.6 \]

(See Figure 5.8)

\[ 2\omega_n = \frac{B + K_{hw} k_v}{M} \]

\[ \therefore K_v = \frac{(2\omega_n M - B)}{K_{hw}} \]

Velocity Feedback Gain
\[ P.O. = \frac{P_i - x_{ss}}{x_{ss}} \times 100\% \]

Spec 3) \( \Rightarrow \) D.C. Gain = 1

\[ T_c[0] = \frac{(K_{nu}K_{pf}/M)}{(K + K_{nu}K_i)/M} \]

\[ T_c[0] = \frac{K_{nu}K_{pf}}{K + K_{nu}K_i} = 1 \]

\[ \Rightarrow K_{nu}K_{pf} = K + K_{nu}K_i \]

\[ K_{pf} = \frac{(K_{nu}K_i + K)}{K_{nu}} \]

\[ K_{pf} \approx K_i + \frac{K}{K_{nu}} \]

Reference Input Gain
From compensated step response, determine percent overshoot, peak time, and steady-state value.

\[ P.O. = \frac{P_i - X_{ss}}{X_{ss}} \times 100\% \]

\[ \zeta = \left[ \frac{\ln^2 \left( \frac{P.O.}{100} \right)}{\pi^2 + \ln^2 \left( \frac{P.O.}{100} \right)} \right]^{1/2} \]

Use the percent overshoot to determine the compensated system's damping ratio.
Use the peak time to determine the compensated system's damped frequency.

\[ T_p = \frac{\pi}{\omega_d}, \quad \omega_d = \frac{2\pi}{T} = \frac{2\pi}{2T_p} = \frac{\pi}{T} \]

The natural frequency can be obtained from \( \gamma \) and \( \omega_d \).

\[ \cos \theta = \gamma, \quad \theta = \cos^{-1}(\gamma) \]

\[ \omega_n^2 = \sigma^2 + \omega_d^2 = (\gamma \omega_n)^2 + \omega_d^2 \]

\[ (1-\gamma^2) \omega_n^2 = \omega_d^2 \quad \Rightarrow \quad \omega_n^2 = \frac{\omega_d^2}{(1-\gamma^2)} \]

\[ \omega_n = \frac{\omega_d}{\sqrt{1-\gamma^2}} \]