On the quantum-channel capacity for orbital angular momentum-based free-space optical communications

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Inspired by recent demonstrations of orbital angular momentum-(OAM)-based single-photon communications, we propose two quantum-channel models: (i) the multidimensional quantum-key distribution model and (ii) the quantum teleportation model. Both models employ operator-sum representation for Kraus operators derived from OAM eigenkets transition probabilities. These models are highly important for future development of quantum-error correction schemes to extend the transmission distance and improve date rates of OAM quantum communications. By using these models, we calculate corresponding quantum-channel capacities in the presence of atmospheric turbulence.

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Recently, the single-photon optical communication that carries orbital angular momentum (OAM) has generated considerable interest due to its high potential for improving the photon efficiency and security for multidimensional quantum-key distribution (MQKD) [1–3]. Atmospheric turbulence represents the predominant detrimental effect in OAM-based free-space optical (FSO) communications. The Kolmogorov atmospheric turbulence effects have been investigated to evaluate the turbulent aberrations on beams with Laguerre–Gauss (LG) modes [1]. In addition to LG beams, the pure vortex beams exhibiting uniform amplitude within the transmitter aperture have been studied to enable atmospheric laser quantum communication systems [2]. The transmitted information rate over the atmospheric turbulence channels can be estimated by developing the transition model of OAM states of individual photons [1,2]. An analytical model describing the transition probabilities among OAM eigenkets propagated over the slant path atmospheric turbulence has been introduced in [3], while the amplitude damping model for the depolarizing quantum channel is discussed with respect to OAM states [4] and channel measurements [5].

In contrast to the single-photon communication system previously described, for which the capacity of the classical Shannon channel has been calculated in [1], we evaluate the influence of atmospheric turbulence on the information rate based on concepts of quantum-information theory instead. By using the transition probabilities of OAM eigenkets, two quantum-channel models have been proposed; the first one is suitable for MQKD applications, and the second one is suitable for quantum-teleportation studies. The Holevo–Schumacher–Westmoreland (HSW) theorem [6] is then used to determine the OAM quantum-channel capacities for these two models in the presence of atmospheric turbulence.

A single photon containing $m\hbar$ units of OAM can be denoted by eigenket $|m\rangle$, where $m = -L, \ldots, 0, \ldots, L$ For $L = 1$, the eigenkets are given by

$$|+1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad |0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad |-1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1}$$

An arbitrary superposition photon quantum state can be represented in terms of OAM eigenkets as follows

$$|\psi_{in}\rangle = \sum_{m=-L}^{L} \alpha_m |m\rangle, \qquad \text{s.t.} \sum_{m=-L}^{L} \alpha_m^2 = 1. \tag{2}$$

Clearly, the superposition state $|\psi_{in}\rangle$ lives in a $(2L+1)$-dimensional Hilbert space.

It follows that the density operator of our quantum-optical communication system [2] can be represented by $\rho_s = \sum_m p_m |\psi_m\rangle\langle\psi_m|$, where p_i describes the probability that a member of the ensemble has been prepared in state $|\psi_m\rangle$. For the simplest case of $L = 1$ and $|\psi_m\rangle = |m\rangle$, the density operator is given by

$$\rho_s = \begin{bmatrix} p_{+1} & 0 & 0 \\ 0 & p_0 & 0 \\ 0 & 0 & p_{-1} \end{bmatrix}. \tag{3}$$

When the baseket $|m\rangle$ is transmitted over the atmospheric turbulence channel, it can be detected on the receiver side as baseket $|n\rangle$ $(n \neq m)$; $n = -L, \ldots, 0, \ldots, L$. To evaluate the quantum-channel capacity in the presence of atmospheric turbulence, we take the first step to construct a corresponding quantum-channel model.

The transition probability from an OAM quantum state $|m\rangle$ to OAM eigenket $|n\rangle$, after propagation through the free-space atmospheric turbulence, is derived in [2,4] as

$$p_{m,n} = \langle s_\Delta \rangle = \frac{1}{\pi} \int_0^1 \mathrm{d}\rho\rho \int_0^{2\pi} \mathrm{d}\theta \times \exp\left[-3.44\left(\frac{D}{r_0}\right)^{5/3}\right]\left(\rho \sin\frac{\theta}{2}\right)^{5/3} \cos\Delta\theta, \tag{4}$$

where $\Delta = |n - m|$, $\rho = r/R$, $D = 2R$. The r_0 and R stand for the Fried coherence diameter and the radius of the transmitter telescope, respectively. The transition probabilities given by Eq. (4) are applicable to the vortex beams, whose electric field can be represented as [2]

$$E(r) = E_0 W\left(\frac{r}{R}\right) e^{im\theta}, \tag{5}$$

where E_0 denotes the amplitude of a (spatially uniform) electric field, $W(x)$ stands for the aperture function that is equal to one for $|x| \le 1$ and zero otherwise [2].

In quantum-information theory (see [5,6]), a quantum channel can be described by the transformation of an input density matrix ρ_s to the output density matrix ρ_s'. This transformation, described by the quantum operation U, can be represented as the following mapping

$$\rho_s \overset{U}{\rightarrow} \rho_s'. \tag{6}$$

Clearly, the super-operator U cannot be unitary due to the decoherence effects. However, the total evolution operator of quantum system and environment can be represented by unitary operator $U_{s,E}$. Without loss of generality, let us assume that the environment E is initially in a pure state $|0_E\rangle$. Hence, the expression for super-operator under this initial condition can be written as

$$\xi(\rho_s) = \mathrm{Tr}_E U_{s,E}(\rho_s \otimes |0_E\rangle\langle 0_E|) U_{s,E}^{\dagger}, \tag{7}$$

where the partial trace $\mathrm{Tr}_E(\cdot)$ is taken with respect to the environmental degrees of freedom. Consider Eq. (7) as a completely positive linear transformation acting on the density matrix, which can be reconstructed as

$$\xi(\rho_s) = \sum_i E_i \rho_s E_i^{\dagger}. \tag{8}$$

The corresponding Kraus operators E_i in Eq. (8) satisfy the completeness relationship such that

$$\sum_i E_i^{\dagger} E_i = I. \tag{9}$$

Equation (9) is equivalent to the trace-preserving requirement given by $\mathrm{Tr}[\xi(\rho_s)] = 1$. On the other hand, any set of operators E_i satisfying Eq. (9) is applicable to Eq. (8), and thereafter gives rise to a valid noisy channel with Eq. (7) as its super-operator. It has been shown in [6] that Eq. (9) is related to the Schrödinger evolution of the density matrix.

Below we describe two relevant channel models: (i) the model suitable for MQKD studies in which $|\psi_m\rangle = |m\rangle$, and (ii) superposition state model suitable for study of OAM-based teleportation systems.

For the first model, in the presence of atmospheric turbulence, the Kraus operator $E_{m,n}$ performing the transformation from $|m\rangle$ to $|n\rangle$ can be constructed according to the transition probabilities in Eq. (4). The simplest case of $L = 1$ is depicted in Fig. 1(a). A transformation of OAM baseket $|+1\rangle$ to $|-1\rangle$ can be represented as $U_{+1,-1} = |-1\rangle\langle +1|$. Therefore, the turbulent quantum free-space channel model contains in total $(2L + 1)^2$ Kraus operators.

The resulting density matrix upon propagation through the turbulent quantum channel can now be expressed as

$$\mathbf{V}(\rho_s) = \sum_{m,n} p_{m,n} U_{m,n} \rho_s U_{m,n}^{\dagger}. \tag{10}$$

The Kraus operator $E_{m,n}$ now becomes

$$E_{m,n} = \sqrt{p_{m,n}} U_{m,n}. \tag{11}$$

Notice that partial trace over the environment qubit has been omitted due to the fact that a mixing process [7] can be described either with or without a fictitious environment. In other words, $\mathbf{V}(\rho_s)$ is equivalent to the density operator $\xi(\rho_s)$ after tracing out the environment.

The second model suitable for the study of the transmission of superposition states over turbulent channels can be called the quantum-teleportation model, which, for $L = 1$, is illustrated in Fig. 1(b). The Kraus operator $E_{i,j,k}$ is given by

$$E_{i,j,k} = \sqrt{p_{+1,i} p_{0,j} p_{-1,k}} (|i\rangle\langle +1| + |j\rangle\langle 0| + |k\rangle\langle -1|),$$
$$i, j, k \in \{-1, 0, +1\}. \tag{12}$$

Clearly, the total number of Kraus operators in this model is $(2L + 1)^{2L+1}$.

By employing the HSW theorem [6], the quantum-channel capacity for both models can be calculated as follows:

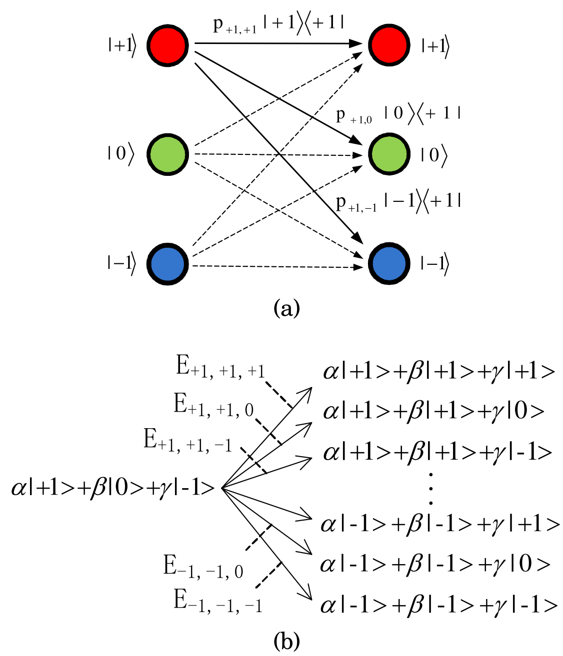

Fig. 1. (Color online) The OAM quantum-channel models for $L = 1$: (a) MQKD model ($|\psi_m\rangle = |m\rangle$) and (b) quantum teleportation model.

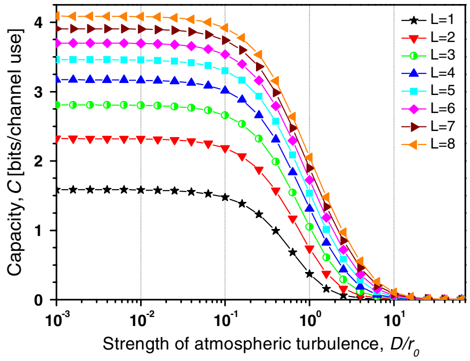

Fig. 2. (Color online) Quantum-channel capacity for MQKD model (i) against the atmospheric turbulence strength (in terms of telescope diameter D over Fried's coherence diameter r_0).

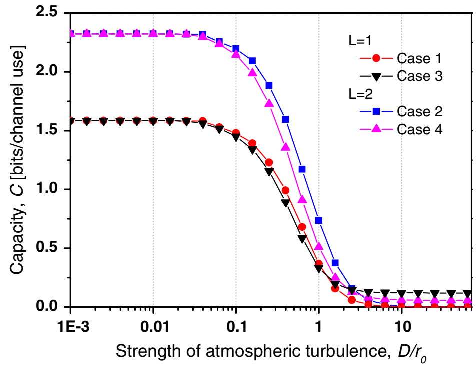

Fig. 3. (Color online) QCCs against the atmospheric turbulence strength for various superposition states using the quantum teleportation model (ii).

$$C(\rho_s) = \max_{\{p_j,\rho_j\}}\left\{S[\mathbf{V}(\rho_s)] - \sum_j p_j S[V(\rho_j)]\right\}, \quad \rho_s = \sum_j p_j \rho_j, \tag{13}$$

where the maximization is performed over all ensembles $\{p_j, \rho_j\}$ of possible input states ρ_j. With $S[(U(\rho_s)] = -\mathrm{Tr}(U \log_2 U)$, we denoted the von Neumann entropy [6].

The results of quantum-channel capacity (QCC) with respect to the atmospheric turbulence strength, based on model (i), are depicted in Fig. 2. Notice that these results are in good agreement with experimental classical results reported in [8], which is expected, as the Shannon entropy is the upper bound of the von Neumann entropy. Notice that transition probabilities by Eq. (4) are not valid in the strong turbulence regime, while the proposed model is valid for arbitrary turbulence strength. In a strong turbulence regime, one may use the model described in [9] to determine the transition probabilities (once this model get experimentally verified).

As the maximization with respect to $(2L+1)^{2L+1}$ transition probabilities is numerically intractable, in Fig. 3 we provide the selected results for model (ii) for the following cases:

- Case 1 ($L = 1$) corresponds to QCC of transmitting states $|+1\rangle$, $|0\rangle$, and $|-1\rangle$, selected at random.
- Case 2 ($L = 2$) corresponds to QCC of transmitting $|+2\rangle$, $|+1\rangle$, $|0\rangle$, $|-1\rangle$, and $|-2\rangle$, selected at random.
- Case 3 ($L = 1$) corresponds to QCC of transmitting:

$$\frac{|+1\rangle + |-1\rangle}{\sqrt{2}}, \ \frac{|+1\rangle + 2|0\rangle - |-1\rangle}{\sqrt{6}}, \text{and} \ \frac{-|+1\rangle + |0\rangle + |-1\rangle}{\sqrt{3}}.$$

- Case 4 ($L = 2$) corresponds to QCC of transmitting:

$$\frac{|1\rangle + |-2\rangle}{\sqrt{2}}, \ \frac{|2\rangle + 2|1\rangle - |-2\rangle}{\sqrt{6}}, \ \frac{|2\rangle + |1\rangle + 3|0\rangle + |-2\rangle}{\sqrt{12}},$$
$$\frac{|2\rangle - |1\rangle + |0\rangle + 4|-1\rangle + |-2\rangle}{2\sqrt{5}}, \frac{-|2\rangle + |1\rangle - |0\rangle + |-1\rangle + |-2\rangle}{\sqrt{5}}.$$

For both $L = 1$ and $L = 2$, the capacity of Cases 1 and 2 are higher than those in Cases 3 and 4, in which transmitted quantum states are the superpositions of basekets $\{|-L\rangle, \ldots, |L\rangle\}$. However, when the channel becomes completely corrupted, the capacities of Cases 3 and 4 tend to be constant non-zero values, while Cases 1 and 2 tend to zero. Namely, the saturation turbulence regime will cause equiprobable transition from any baseket to any other baseket in $\{|-L\rangle, \ldots, |L\rangle\}$, which represents the worst case scenario for 1 and 2, but not for Cases 3 and 4.

In summary, we presented two quantum-channel models suitable for OAM-based free-space optical-quantum communication. The first model is suitable for MQKD, and the second one is suitable for quantum-teleportation applications. By using the transition probabilities of OAM eigenkets, the quantum-channel models are developed by exploiting the operator-sum representation. The proposed quantum-channel models are of crucial importance for future study of quantum-error correction for vortex beam based free-space quantum-optical communications, in order to extend the transmission distance, improve tolerance to atmospheric turbulence, and enable high data-transmission rates. In addition, by using the aberration correction method described in [10], it is possible to improve the quantum-channel capacity in the strong turbulence regime.

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