Iteratively Decodable Codes from Orthogonal Arrays for Optical Communication Systems

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***Abstract*— A novel family of low-density parity-check codes is proposed based on orthogonal arrays. Codes from this family have high code rate, girth of at least six, large minimum distance, and significantly outperform the error correction schemes based on turbo product codes proposed for optical communication systems.**

***Index Terms*— Forward error correction, low-density parity-check codes, orthogonal arrays, optical communications.**

I. Introduction

THE STATE of the art optical communication systems standardized by the ITU-T employ different Bose-Chaudhuri-Hocquenghem (BCH) codes [1]: (4359,4320) BCH-3 code in the ITU-T G.707 standard, (255,238) Reed-Solomon (RS) code in G.709 standard, and (255, 239) RS code in G.975 standard [2]. Recently [3-9] iteratively decodable codes, turbo [3] and low-density parity-check (LDPC) [4-9] codes, have generated great research attention in the coding community. An iterative LDPC code proposed by Chung *et al.* has been shown to achieve a performance as close as 0.0045 dB to the Shannon limit [4], on additive white Gaussian noise (AWGN) channel. Inherent low complexity decoder based on the sum-product algorithm (SPA) opens up avenues for its use in different high speed applications, such as optical communications [5-6]. In a series of articles [5-6] we showed that error performance and decoder hardware complexity offered by turbo codes can be matched and outperformed by iteratively decodable LDPC codes.

In this paper a novel class of LDPC codes based on *orthogonal arrays* (OAs) is proposed. These codes have large minimum distance, girth of at least six, low complexity iterative decoding algorithm, excellent bit-error rate (BER) performance, and the construction algorithm is very simple. In order to make a realistic assessment of the codes performance in a long-haul optical communication system, an advanced simulator was developed [16] that is able to catch most important transmission impairments such as amplified spontaneous emission (ASE) noise, pulse distortions due to fiber nonlinearities, chromatic dispersion, cross-talk effects, and inter-symbol interference. The LDPC code (4096,3555) based on 1-OA(64^2,65,64,2) with an overhead of only 14.9% outperforms the best turbo code proposal for optical communication systems [3] with 24.6% of overhead by 0.4 dB at 10^{-10}. It also outperforms an affine geometry [AG(3,3^2)] based LDPC(7371,6553) code of comparable rate by 0.5 dB at 10^{-10}, and a projective geometry [PG(3,3^2)] based LDPC(4745,4344,0.915) by 0.4 dB. The construction algorithm of iteratively decodable codes from orthogonal arrays is significantly simpler than that based on finite geometries [8], MacNeish-Mann theorem [5], or RS based LDPC codes [7]. Apart from excellent coding gain, comparable or better than that of finite geometries codes, these LDPC designs offer larger number of codes of code rate above 0.8; the code rate limitation is imposed by available high-speed electronics.

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II. Codes from Orthogonal Arrays

The codes proposed in this paper are based on the theory of orthogonal arrays [10-15]. An *orthogonal array* of size *N*, with *k* constraint, *q* levels, strength *t*, and index λ, denoted as λ-OA(*N*,*k*,*q*,*t*), is defined as a *k*x*N* matrix *A* with entries from a set of $q(\geq 2)$ elements such that any *t*x1 column vector in *t*x*N* submatrix of *A* is contained λ times.

Example 1: The 1-OA(9,4,3,2) is given below

$$\begin{array}{ccccccccc} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \end{array}$$

Any two rows (observed in vertical pairs), in the example above, contain each ordered pair of elements exactly once. The orthogonal arrays of strength 2 and index unity are denoted by OA(*N*,*k*,*q*). If in OA(*N*,*k*,*q*) $N = q^2$, the problem of existence of OA is equivalent to the existence of *k*-2 mutually orthogonal Latin squares (MOLS) [5]. In Example 1, the first two rows label the positions in the MOLS, the first row corresponding to the row index, and second row corresponding to the column index. The third and fourth rows represent the entries of corresponding squares. Therefore, the following two MOLS are obtained from 1-OA(9,4,3,2):

$$L_1 = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \texttt{ and } L_2 = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}.$$

Recently, in [5], we have shown that there exists one-to-one correspondence between a MOLS and the parity-check matrix of an LDPC code. When $N \neq q^2$, the OA of strength 2 and unit index may be considered as generalization of a complete system of MOLS, since the squares obtained from OA are not strictly speaking MOLS. Moreover, Shrikhande [12] has shown that the generalized Hadamard matrices, affine resolvable balanced incomplete block designs (BIBDs), and group-divisible designs are related to the orthogonal arrays of strength 2. A BIBD or 2-design [10-12], denoted as 2-(v,k,λ), is a collection of *k*-subsets (blocks) of a *v*-set *V* such that every 2-subset of *V* is contained in exactly λ

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blocks. In our constructions, however, every t-subset of V is contained in no more than λ blocks, and is denoted by $2-(v,k,\{0,1,\ldots,\lambda\})$, referred to as λ-configuration [10]. Codes based on OAs with $\lambda > 1$ have short cycles, and therefore we will restrict our attention to the case $\lambda = 1$, since the corresponding bipartite graph has girth at least 6. For example, 1-configuration from 1-OA(9,4,3,2) is composed of the following blocks: {1,6,8}, {3,5,7},{2,4,9}, {1,5,9},{2,6,7} and {3,4,8}. The 1-configuration is derived from Example 1 by writing down the labels of elements {0,1,2} in the third and fourth rows. The first two rows are used as two-dimensional labels converted into one-dimensional by the following linear transformation:

$$l(i,j) = i(k-1) + j + 1; i,j = 0,1,...,k-2. \tag{1}$$

The first block {1,6,8} is derived from labels of 0-element in the third row: {(0,0),(1,2),(2,1)}⇔{1,6,8}. The corresponding parity-check matrix of an LDPC code is

$$H = \begin{bmatrix} 1&0&0&0&0&1&0&1&0\\ 0&0&1&0&1&0&1&0&0\\ 0&1&0&1&0&0&0&0&1\\ 1&0&0&0&1&0&0&0&1\\ 0&1&0&0&0&1&1&0&0\\ 0&0&1&1&0&0&0&1&0 \end{bmatrix}.$$

The labels in 1-configuration blocks describe the positions of ones in corresponding rows of parity-check matrix. For example, the first block {1,6,8} corresponds to the first row in *H*-matrix, 100001010.The girth (the shortest cycle in Tanner graph corresponding to *H*-matrix) in this particular example is 8. The parity-check matrix $H = (h_{i,j})$ is defined by:

$$h_{i,j} = \begin{cases} 1, & \text{if the ith block contains the jth element} \\ 0, & \text{otherwise} \end{cases}$$

Therefore, a parity check matrix of an LDPC code can be readily constructed from orthogonal arrays of strength two and unity index. For example, Addelman and Kempthorne proposed [15] a method to design orthogonal arrays $(2q^n, 2(q^n-1)/(q-1)-1, q, 2)$; Rao [13] considered the constructions of hypercubes from PG(t,q) and Bush [11] proposed a method to construct $1-\mathrm{OA}(q^t, q+1, q, t)$. Each of those constructions is directly applicable in design of LDPC codes as described above. In the rest of the section an LDPC code is designed based on $1-\mathrm{OA}(q^t, q+1, q, t)$ construction due to Bush [11], as an illustrative example. The construction is based on a Galois field of a prime power $q = p^s$, where p is a prime, and s is an integer, $s \geq 1$. Let $\alpha_0 = 0, \alpha_1, \alpha_2, \ldots, \alpha_{q-1}$ be the elements of GF(q). Consider the following q^t polynomials

$$y_j(x) = a_0 + a_1 x + a_2 x^2 + ... + a_{t-1} x^{t-2} + a_{t-1} x^{t-1}, \tag{2}$$

with coefficients from GF(q). An (q+1) by q^t OA is formed by filling the (i,j)-th cell with symbol u determined from

$$y_j(\alpha_i) = \alpha_u. \tag{3}$$

The $(q+1)-$th row is formed by putting the symbol u into the columns corresponding to the polynomials with leading coefficient α_u. Notice that Bush construction for t=2, construction due to Rao [13], and that affine geometries from [8] are related. The code rate is determined by $R = (N - rank(H))/N$, where N is the code length, and is lower bounded by $1-(k-2)q/N$. To keep code rate above R_0 (e.g., 0.8) $k-k_s-2$, $k_s = \lfloor 2 + (1-R_0)N/q \rfloor$, rows in OA should be disregarded. Moreover, the number of cycles of length 6 will be reduced, which might result in better BER performance compared to AG based codes as shown in Section III.

Example 1 (cont.): Let us explain how the 1-OA(9,4,3,2) from Example 1 was created. The elements from GF(3) are 0,1, and 2. Corresponding polynomials from (2) are: $y_1 = 0, y_2 = 1, y_3 = 2, y_4 = x, y_5 = 1+x, y_6 = 2+x, y_7 = 2x, y_8 = 1+2x, y_9 = 2+2x$.

The first three rows in Example 1 are obtained by calculating the values of polynomials for 0, 1, and 2, respectively (operations are performed in GF(3)). The last row is formed by reading off the leading coefficients in $y_1, y_2, \ldots, y_9$.

Example 2: The elements of GF(4) (displayed as 2-tuples) are: $\alpha_0 = [00], \alpha_1 = [10], \alpha_2 = [01], \alpha_3 = [11]$. Corresponding OA(16,5,4,2), obtained by applying (2) and (3), is:

0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
0	1	2	3	1	0	3	2	2	3	0	1	3	2	1	0
0	1	2	3	2	3	0	1	3	2	1	0	1	0	3	2
0	1	2	3	3	2	1	0	1	0	3	2	2	3	0	1
0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3

and corresponding 1-configuration, after the affine transformation (1), is

$$B = \begin{bmatrix} 1&4&2&3&1&3&4&2&1&2&3&4\\ 7&6&8&5&8&6&5&7&6&5&8&7\\ 12&9&11&10&10&12&11&9&11&12&9&10\\ 14&15&13&16&15&13&14&16&16&15&14&13 \end{bmatrix}.$$

In *B*, each column represents a block from an 1-configuration. Corresponding parity-check matrix of an LDPC code is

$$H = \begin{bmatrix} 1&0&0&0&0&0&1&0&0&0&0&1&0&1&0&0\\ 0&0&0&1&0&1&0&0&1&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&1&0&0&1&0&1&0&0&0\\ 0&0&1&0&1&0&0&0&0&1&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&1&0&1&0&0&0&0&1&0\\ 0&0&1&0&0&1&0&0&0&0&0&1&1&0&0&0\\ 0&0&0&1&1&0&0&0&0&0&1&0&0&1&0&0\\ 0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&1\\ 1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\ 0&1&0&0&1&0&0&0&0&0&0&1&0&0&1&0\\ 0&0&1&0&0&0&0&1&1&0&0&0&0&1&0&0\\ 0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0 \end{bmatrix}.$$

As above, the columns in *B* give the positions of ones in rows of *H*. The first two and last two polynomials used in the construction of OA are given as an illustration, respectively as: $y_1 = \alpha_0 + \alpha_0 x, y_2 = \alpha_1 + \alpha_0 x, \ldots, y_{15} = \alpha_2 + \alpha_3 x, y_{16} = \alpha_3 + \alpha_3 x$.

There is also a possibility to establish the connection between an orthogonal array and 1-configuration on the following way. Denote the positions in every row of an orthogonal array from 1 to *N*. The positions of zero elements in ith row determine the block $B_{i,0}$, the positions of ones determine the block $B_{i,1}$, and so on, the positions of element $q-1$ determine the block $B_{i,q-1}$. For example, the positions of zeros in third row of Example 2 determine the block {1,7,12,14}, the positions of ones the block {2,8,11,13}, the positions of element 2 the block {3,5,10,16}, and finally the positions of element 3 the block {4,6,9,15}. Those four blocks are identical to the first four columns of matrix B.

It has been shown in [14] that existence of orthogonal arrays $\mathrm{OA}(N_i, k_i, q_i, t)(i = 1,2,\ldots,I)$ implies the existence of the orthogonal array $\mathrm{OA}(N,k,q,t)$, where $N = N_1 N_2 \cdots N_I, q = q_1 q_2 \cdots q_I$, and $k = \min(k_1, k_2, \ldots, k_I)$. Therefore, using the concept of product of orthogonal arrays the construction above can be extended to support larger number of LDPC codes of code rate above 0.8.

III. Simulation Results

A wavelength-division multiplexing (WDM) system with 40Gb/s bit-rate per channel and a channel spacing of 100 GHz is considered. It is assumed that the observed channel is

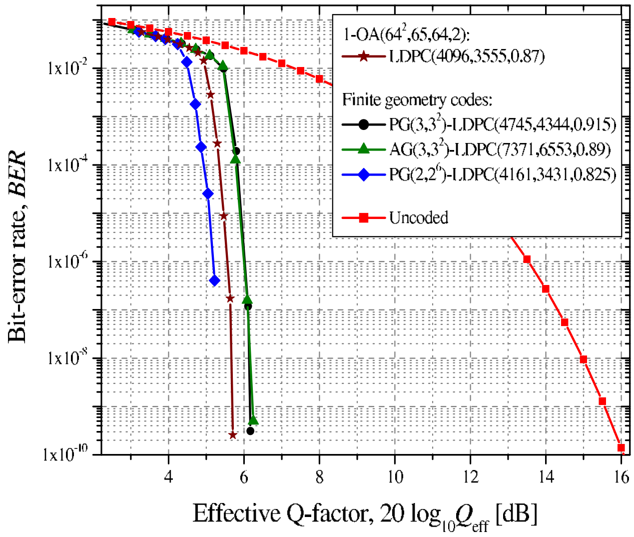

Fig. 1. BER performance of LDPC codes at 40 Gb/s in 25th iteration

located at 1552.524 nm and that there exists a non-negligible interaction with six neighboring channels. The dispersion map is composed of 25 spans of length *L*=48 km, each span consisting of 2*L*/3 km of D+ fiber followed by *L*/3 km of D- fiber. The fiber parameters may be found in [6]. The pre-compensation of -320 ps/nm and corresponding post-compensation are also used. The erbium-doped fiber amplifiers (EDFAs) are deployed after every fiber section. The optical filter of bandwidth $2R_l$ and electrical filter of bandwidth $0.65R_l$ are used in simulation, where R_l denotes the line rate (the bit rate over code rate). The average launched power is set to 0 dBm, MZ modulator extinction ratio to 14 dB and the carrier-suppressed RZ (CSRZ) modulation format is observed. The Q-factor is additionally decreased by noise loading. The simulator described in our previous paper [16] is used to create samples fed to iterative decoder.

Fig. 1 plots the BER results obtained from Monte Carlo simulations for a 1-OA(64^2,65,64,2) based iteratively decodable (4096,3555,0.8679) code (with *H*-matrix of size 1008x4096, row weight 64, and column weight 16) against finite geometry codes. The efficient realization SPA proposed in [9] is employed in simulations, which allows additional 0.5 dB improvement in coding gain compared to the min-sum approximation of SPA implemented in [16]. The LDPC(4096,3555,0.8679) code of redundancy 14.9% from OA outperforms the best turbo code proposal [3] for optical transmission of significantly higher redundancy 24.3% by 0.4 dB at 10^{-10}. The same code outperforms affine geometry based LDPC(7371,6553,0.89) of comparable rate by 0.5 dB, and projective geometry based LDPC(4745,4344,0.915) by 0.4 dB at BER of 10^{-10}. PG(2,2^6) based LDPC(4161,3431,0.825) code outperforms the OA based code, however it is of significantly higher redundancy (21.2%). LDPC(4096,3555,0.8679) code from 1-OA(64^2,65,64,2) provides the coding gain of 10.3 at BER of 10^{-11}, the largest ever reported coding gain for optical transmission for this overhead.

IV. Conclusion

We proposed a novel class of iteratively decodable codes, with high code rate, large minimum distance, girth at least six, and with very simple construction algorithm, based on combinatorial objects known as orthogonal arrays. The proposed LDPC codes perform very well in the presence of ASE noise, fiber nonlinearities, chromatic dispersion, and inter-symbol interference. The LDPC code on 1-OA(64^2,65,64,2) with an overhead of 14.9% outperforms the best turbo code proposal for optical communication systems [3] with 24.6% of overhead by 0.4 dB at 10^{-10}. Notice also that the number of codes from affine planes or projective planes with code rates larger than 0.8 is rather limited [8]. Since the large decoding delay cannot be tolerated in high-speed transmission, the LDPC codes of lengths smaller than 5000 are highly desirable. Those two requirements reduce the number of codes from affine planes AG(2,2^s) applicable to optical communications to only one [8].

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