We derive the elastic collision rate for a harmonically trapped Fermi gas in the extreme unitarity limit where the s-wave scattering cross section is \( \sigma(k) = 4\pi/k^2 \), with \( \hbar k \) the relative momentum. The collision rate is given in the form \( \Gamma = \gamma I(T/T_F) \) —the product of a universal collision rate \( \gamma = k_B T_F/(6\pi\hbar) \) and a dimensionless function of the ratio of the temperature \( T \) to the Fermi temperature \( T_F \). We find that \( I \) has a peak value of \( I = 4.6 \) at \( T/T_F = 0.4 \), \( I = 8.2 (T/T_F)^2 \) for \( T/T_F < 0.15 \), and \( I = 2(T/F)^2 \) for \( T/F > 1.5 \). We estimate the collision rate for recent experiments on a strongly-interacting degenerate Fermi gas of atoms.

The final part of the paper defines a hydrodynamic parameter \( \phi = \Gamma/\omega_z \), the ratio of unitarity-limited collision rate \( \Gamma \) to the transverse oscillation frequency \( \omega_z \) of atoms in the trap. We calibrate \( \phi \) by observing the threshold for hydrodynamic expansion of a strongly interacting Fermi gas as a function of evaporation time [1]. We estimate \( \phi \) for several recent experiments on strongly interacting Fermi gases [1,4,5].

II. CALCULATING THE COLLISION RATE

We consider a simple model that assumes that s-wave scattering is dominant [15]. In this case, the collision cross section takes the form

\[
\sigma(k) = \frac{4\pi a_s^2}{1 + k^2 a_s^2},
\]

where \( k \) is the relative wave vector of a colliding pair of spin-up and spin-down fermionic atoms.

In the trap, the average collision rate per particle, \( \Gamma \), is determined from the s-wave Boltzmann equation [16] under the assumption of sufficient ergodicity. We consider the rate for the process in which a spin-up and a spin-down atom of total energy \( E_i = E_1 + E_2 \) collide to produce atoms with total energy \( E_o = E_1 + E_2 \). The effects of Pauli blocking are included for the particles on the outgoing channel, and we assume a 50-50 mixture of atoms in the two spin states. The depletion term in the Boltzmann equation for the particle of energy \( E_4 \) integrated over \( E_i \) to determine the collision rate \( \Gamma \) for either spin state (as a collision inherently includes one atom of each spin). For an energy-independent cross section, the integrated loss rate is then \( N/2 = -\Gamma N/2 \), and

\[
\frac{N}{2} = \frac{M\sigma}{2\pi^2\hbar^3} \int d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4 D(\epsilon_{\text{min}}) \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \times (1-f_1)(1-f_2)f_3f_4,
\]

where \( \Gamma \) is the number of collisions per second per atom, \( N \) is the total number of atoms in the trap, and \( M \) is the atomic mass. Here, \( D(\epsilon_{\text{min}}) \) is the density of states evaluated at the energy \( \epsilon_{\text{min}} = \min(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \), \( f_i = 1/(g_i + 1) \) is the occupation number with \( g_i = \exp((\epsilon_i - \mu)/k_B T) \), and \( \mu \) is the chemical potential [17]. At zero temperature, the chemical potential is given by the Fermi energy \( \mu(0) = E_F = (3N)^{1/3} \hbar \omega_z \).
\(= k_B T_F \), where \(\tilde{\omega} = (\omega_\perp^2 \omega_z)^{1/3} \) with \(\omega_\perp\) and \(\omega_z\), the transverse and axial trap oscillation frequencies of a cylindrically symmetric trap.

For fermions, the collision cross section of Eq. (1) with \(|ka_s| < 1\) is \(\sigma = 4\pi a_s^2\), i.e., half that for indistinguishable bosons. We begin by determining \(\Gamma\) for this case.

### A. Energy-independent cross section

The integrand in Eq. (2) is readily shown to be symmetric under the interchange of all four particle labels. Hence, without loss of generality, we multiply the integrand by 4, and take \(\epsilon_{\text{min}} = \epsilon_1\) and \(D(\epsilon_{\text{min}}) = \left(\epsilon_1^2/(2\hbar^2\tilde{\omega})\right) \theta_{21} \theta_{41}\), where \(\theta_{21} = \theta(\epsilon_2 - \epsilon_1)\) is a unit step function.

It is useful to write the collision rate as the product of a natural collision rate \(\gamma_{\text{UL}}\), which depends on the trap parameters, and a dimensionless integral \(I_{\text{UL}}(T/T_F)\), which describes the temperature dependence,

\[
\Gamma = \gamma_{\text{UL}} I_{\text{UL}}(T/T_F). \tag{3}
\]

We take the natural collision rate to be the classical collision rate at \(T = T_F\),

\[
\gamma_{\text{UL}} = \frac{NM \tilde{\omega}^3 \sigma}{4\pi^2 k_B T_F}. \tag{4}
\]

Note that the rate is 1/4 of that obtained in a spin-polarized Bose gas. With this choice, \(I_{\text{UL}}\) becomes

\[
I_{\text{UL}}(T/T_F) = 144 \int_0^\infty \int_0^\infty dx_1 dx_2 dx_3 x_1^2 f(x_1 + x_2) f(x_1 + x_3)
\]

\[
\times \left[1 - f(x_1)\right] \left[1 - f(x_1 + x_2 + x_3)\right]. \tag{5}
\]

Here \(f(x) = \mathcal{I}[g(x) + 1]\), where \(g(x) = \exp[(T_F/T)(x - \mu/\epsilon_F)]\). We assume that for the cases of interest, the trap depth is large compared to \(\epsilon_F\) and \(k_B T_F\).

\(I_{\text{UL}}\) is readily determined by numerical integration using standard results for the chemical potential as a function of \(T/T_F\) [17]. At low temperature, \(T/T_F \ll 0.2\), we find that \(I_{\text{UL}}\) is well fit by \(I_{\text{UL}}(T/T_F) \approx 15 (T/T_F)^2\), which displays the quadratic dependence expected for Pauli blocking in both final states. At high temperature, \(T/T_F > 1.5\), we find the expected temperature dependence, \(I_{\text{UL}}(T/T_F) \approx T/T_F^2\), as shown below. The complete function \(I_{\text{UL}}(T/T_F)\) is plotted in Fig. 1(a). The maximum value, \(I_{\text{UL}} \approx 1.3\), occurs for \(T/T_F = 0.5\). This demonstrates that \(\gamma_{\text{UL}}\) is essentially the maximum collision rate.

### B. Unitarity-limited cross section

To include the energy dependence of the cross section, we adopt the notation of Ref. [16], and make the replacement

\[
\sigma D(\epsilon_{\text{min}}) = \frac{2\pi M}{(2\pi \hbar)^3} \int_{\epsilon(x) < \epsilon_{\text{min}}} dx \int_{P_>(x)} P_<(x) P_\sigma(q), \tag{6}
\]

where \(F(x, x_m)\) determines the energy-dependent cross section \(\sigma(q) = 4\pi/q^2\) in units of \(4\pi/k_F^2\). The arguments of \(F(x, x_m)\) are \(x = \epsilon/\epsilon_F\) and \(x_m = \epsilon_{\text{min}}/\epsilon_F\). As in Eq. (5), we take \(x_m = x_1\) without loss of generality.

For a harmonic potential,
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\[
F(x,x_m) = \frac{16}{\pi \sqrt{2} x_m} \int_0^1 \frac{du \, u^2}{\sqrt{1-u^2}} \ln \left[ \frac{\chi_+ (\alpha, u)}{\chi_- (\alpha, u)} \right],
\]

where \( \chi_+ (\alpha, u) = A(\alpha, u) + B(\alpha, u) \).

We can simplify the form of the integrals in Eq. (9) by transforming from coordinates \( \{x_1, x_2, x_3, x_4\} \) into coordinates \( \{w, y, z, s\} \), where \( w = x_1, y = (x_2 + x_3)/2x_1, z = (x_3 - x_2)/2x_1, \) and \( s = 1 - u^2 \).

We can simplify the form of the integrals in Eq. (9) by transforming from coordinates \( \{x_1, x_2, x_3, x_4\} \) into coordinates \( \{w, y, z, s\} \), where \( w = x_1, y = (x_2 + x_3)/2x_1, z = (x_3 - x_2)/2x_1, \) and \( s = 1 - u^2 \).

The spin-up atoms at position \( x_1 \) agree with the numerical results for \( I_{UL}(T/T_F) \) at \( 4.6 \) for \( T \geq T_F \) in the high-\( T/T_F \) limit, where \( N = \frac{3\lambda N^2}{2\pi} \). The results are plotted in Fig. 2.

C. Comparison with analytic high-\( T/T_F \) results

We can model the numerical results for the temperature dependence of the rates by calculating the collision rate \( \Gamma_{HT} \) for \( T \geq T_F \) by the phase-space \( s \)-wave Boltzmann equation \( \chi_+ (\alpha, u) = A(\alpha, u) + B(\alpha, u) \).

We can simplify the form of the integrals in Eq. (9) by transforming from coordinates \( \{x_1, x_2, x_3, x_4\} \) into coordinates \( \{w, y, z, s\} \), where \( w = x_1, y = (x_2 + x_3)/2x_1, z = (x_3 - x_2)/2x_1, \) and \( s = 1 - u^2 \).

The spin-up atoms at position \( x_1 \) agree with the numerical results for \( I_{UL}(T/T_F) \) at \( 4.6 \) for \( T \geq T_F \) in the high-\( T/T_F \) limit, where \( N = \frac{3\lambda N^2}{2\pi} \). The results are plotted in Fig. 2.

For the shortest evaporation times shown in Fig. 2, we observe ballistic scaling with \( T = 3T_F \) and \( N = 4 \times 10^3 \). For our trap, the initial aspect ratio \( \lambda = 0.035 \). From Eq. (13), we obtain \( \phi_z = 0.4 \) which then corresponds to approximately collisionless behavior. Therefore, \( \phi = 0.4 \) is, in general, the condition for collisionless behavior for any time scale \( 1/\omega \).

We choose the characteristic time scale to be \( 1/\omega \), and take \( \phi = \Gamma/\omega \) where \( \omega = \omega_c \). The oscillation frequencies of atoms in the trap. These are also the natural time scales for ballistic expansion, where the size of the cloud scales as \( \sqrt{1 + \omega^2 t^2} \) in each direction. In the unitarity-limited regime, and for a cylindrically symmetric trap with elongation parameter \( \lambda = \omega_c/\omega \), we can write \( \phi_z = \phi_{1/3} \).

\[
\phi_{1/3} = \left( \frac{3\lambda N}{2\pi} \right)^{1/3} \frac{I_{UL}(T/T_F)}{6N},
\]

where \( N \) is the total number of atoms in the 50-50 mixture. Then, \( \phi_z = \phi_{1/3} \).

A. Calibrating \( \phi \)

We now turn to the question of determining the approximate value of \( \phi \) for which the transition between collisionless and collisional behaviors occurs. In Ref. [1], we investigated the anisotropic expansion properties of a strongly interacting, degenerate Fermi gas of \( ^6 \)Li. When released from a highly elongated trap, the originally narrow dimensions of the gas expanded rapidly, while the broad dimension remained largely unchanged—invoking the aspect ratio of the cloud after 1 ms of expansion. We have studied how the observed aspect ratio of the expanded cloud varies with the duration of evaporative cooling. The aspect ratios are measured for a fixed expansion time of 600 \( \mu \)s and are compared to the predictions of ballistic and hydrodynamic expansion. The results are plotted in Fig. 2.

From this figure, we see that the hottest clouds (short evaporation times) expand ballistically, while the coldest clouds (longest evaporation times) expand hydrodynamically. Ballistic expansion is expected in a normal, collisionless gas. Since the rapid transverse expansion extinguishes collisions before the axial distribution can change significantly, the collisional behavior of the expanding gas can be associated with \( \phi_{1/3} \). For the shortest evaporation times shown in Fig. 2, we observe ballistic scaling with \( T = 3T_F \) and \( N = 4 \times 10^3 \). For our trap, the initial aspect ratio \( \lambda = 0.035 \). From Eq. (13), we obtain \( \phi_{1/3} = 0.4 \) which then corresponds to approximately collisionless behavior. Therefore, \( \phi = 0.4 \) is, in general, the condition for collisionless behavior for any time scale \( 1/\omega \).
peratures \(0.08 \leq T/T_F \leq 0.2\). Equation 13 yields \(0.7 \leq \phi_s \leq 3.7\), while \(20 \leq \phi_s \leq 106\). The values of \(\phi_s\) corresponding to our lowest temperatures indicate that the trapped gas is nearly collisionless on the transverse time scale, but collisional on the axial.

The onset of high-temperature superfluidity has been recently predicted in the temperature range \(T/T_F = 0.25 – 0.5\) [6–8]. Since Pauli blocking is ineffective for the unitarity-limited cross section when \(T/T_F \geq 0.25\), it is not clear how collisions in the normal component will affect the formation of this high-temperature superfluid.

For an expanding gas, we cannot make a definitive statement about its collisional nature, even if it were collisionless when trapped, as Pauli blocking may become ineffective as a result of nonadiabaticity in the expansion, deformation of the Fermi surface, or through other effects [19]. The magnitude of these effects, and to what extent they modify \(\phi_s\), remains an open question. We are, therefore, working on experiments which will directly determine if the gas contains a superfluid fraction.

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[14] We take into account a factor of 2 difference between our definition of the collision rate and that of Ref. [12]. We assume the thermal energy to be much smaller than the trap depth while Ref. [12] assumes a truncated distribution to treat evaporation.
[15] We neglect higher-order partial-wave resonances. These sometimes occur near an \(s\)-wave scattering resonance. Carl Williams (private communication).
[18] In the unitarity limit, the mean field rescales the trap radii and maintains a Thomas-Fermi density profile [9].
[19] Henning Heiselberg, Jason Ho, and Wolfgang Ketterle (private communication).