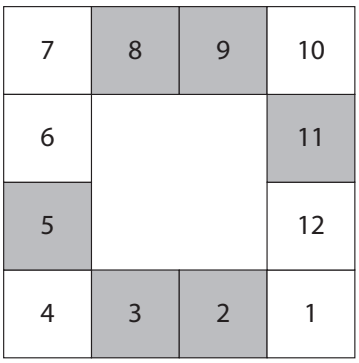
ECE 501b **Homework #4** **Due: 10/22/2012**

1. **Game Strategy:** Consider a multiplayer boardgame that takes place on the following board and with the following rules:



The board contains six squares that are "property" (the shaded squares) and six squares that are not.

1. The game starts with the players on square 1. Each player starts with a fixed number of points (the specific number does not matter).
2. When it is a player's turn, they roll a single, six-sided die and move forward that many squares from their current position.
3. If a player ends their turn on an unowned property square, they may choose to purchase it, provided they have enough points to do so. All property squares have the same cost (the specific cost does not matter).
4. If a player ends their turn on a property square that is already owned, they must pay a penalty to the owner of the property (the specific value of the penalty does not matter). If a player does not have enough points to pay the penalty, they are out of the game.
5. When a player reaches (or passes) square one, they receive a point bonus (the specific value of the bonus does not matter).
6. Play continues until all the players except one have been eliminated. That player is the winner.
7. No turn can end on square 10. If a player's roll would have them finish their turn on square 10, they move to square 4 and end their turn there.
8. If a player's roll would have them finish their turn on squares 6 or 12, they draw a card from a small deck of cards. The card indicates a "special" event that happens to the player. The majority of the cards affect the player's point total only and they end their turn on that square. However, there is a 5% chance of drawing a card that indicates that the player should move to square 1. In this case the player moves to square 1, collects the relevant point bonus, and ends their turn there.

(Some of you may recognize this as a very simplified version of the game *Monopoly*.)

Obviously, you must own properties to win the game. But which ones? The different properties will have different impacts on the probability of your eventual victory.

Use the tools you have learned in this course to develop a simple strategy for this game. Specifically, rank the properties in order of how beneficial they would be to your quest for victory. (Rank them from most beneficial to least beneficial). Be sure to explain the methodology by which you arrived at your strategy. (HINT: Assume the various costs/penalties/etc. are such that the players will circle the board many times before being eliminated. HINT: Think of a player's location as their "state". How are different states connected? Think back to previous homework assignments for inspiration.)

Solution: The key here is to recognize that we can treat a player's motion around the board as a Markov chain (see last homework). Since we're told that the costs/etc. are such that the players will circle the board many times, it doesn't matter where the players start. The long term probability distribution for the different squares will be given by the eigenvector associated with the largest eigenvalue of the transition matrix.

Since winning the game requires having properties that other players will land on, having the properties with the *highest* probabilities of being occupied is a good basic strategy. Therefore, we want to rank the property spaces in terms of their probabilities as predicted by the Markov chain model.

Remember in our earlier discussion, we said the jth column of the transition matrix gives the transition probabilities *from* state j. If we build the transition matrix *just based on the die-roll*, it looks like this:

$$m = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 \end{bmatrix}.$$

This shows that a player in square 1 has a 1/6-chance to end up in any of squares 2-7, and so on.

Now we start folding in the other rules. No turn can end on square 10. Any one who ends there moves to square 4. So, our matrix should have all zeros in row 10, and what is in row 10 should be added to row 4. Using the following commands in MATLAB

```
m(4,:) = m(4,:) + m(10,:);
m(10,:) = zeros(1,12);
```

makes these changes and results in:

$$m = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 \end{bmatrix}.$$

Now we just have to incorporate the effects from the cards in squares 6 and 12. A turn ending in those squares has a 5% chance of actually ending in square 1. So we need to add 0.05 times row 6 and 0.05 times row 12 to row 1. And we need to then multiply rows 6 and 12 by 0.95. Using the following commands in MATLAB

```
m(1,:) = m(1,:) + 0.05*(m(6,:) + m(12,:));
m(6,:) = 0.95*m(6,:);
m(12,:) = 0.95*m(12,:);
```

makes these changes and results in:

$$m = \begin{bmatrix} 1/120 & 1/120 & 1/120 & 1/120 & 1/120 & 1/120 & 21/120 & 21/120 & 21/120 & 21/120 & 21/120 & 21/120 \\ 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 \\ 19/120 & 19/120 & 19/120 & 19/120 & 19/120 & 0 & 0 & 0 & 0 & 0 & 0 & 19/120 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 19/120 & 19/120 & 19/120 & 19/120 & 19/120 & 19/120 & 0 \end{bmatrix}.$$

This is the transition matrix for the game, incorporating all of the relevant rules. Note that each column sums to 1, as we expect. To find the long-time probabilities associated with each square, we look at the eigenvector associated with the largest eigenvalue. Using the following MATLAB command

```
[V D] = eig(m);
```

we learn that the largest eigenvalue is 1 and the associated eigenvector is:

$v = (0.2594, 0.2264, 0.2133, 0.5313, 0.2844, 0.2761, 0.2985, 0.3050, 0.3181, 0, 0.2470, 0.2287)$.

Oops, this isn't properly normalized to be a probability. Dividing by the sum of the elements we get:

$v = (0.0814, 0.0710, 0.0669, 0.1667, 0.0892, 0.0866, 0.0936, 0.0957, 0.0998, 0, 0.0775, 0.0717)$,

which now gives the probability of a player occupying each of the squares in the long-time limit. Looking at the square that are property, and arranging in order of highest to lowest probability, we find: (9,8,5,11,2,3). So, if possible, we should attempt to purchase properties that are early in that list, as those are more likely to generate penalties for our opponents (and income for us).

2. **Quantum Mechanics:** The mathematical language of quantum mechanics is, to a very large degree, linear algebra. Specifically, the following hold:

   1. The state of a quantum mechanical system is represented as a vector in *Hilbert space* (a specific kind of inner-product space). This vector is known as the *wavefunction* for historical reasons. We'll use the symbol ψ to represent the wavefunction.
   2. If the state of the system evolves in time, this can be represented by making the wavefunction time-dependent. That is, we may speak of $\psi(t)$.
   3. For any measurable quantity (e.g. charge, position, velocity, etc.) of the quantum state, there is a corresponding *operator*.
   4. For most operators, the *eigenvectors* of the operator (called the *eigenstates*) can be used as an orthonormal basis for the Hilbert space. If we take $(e_1, \ldots, e_n)$ as the basis, we can write ψ as
   $$\psi = a_1 e_1 + \ldots + a_n e_n.$$
   5. The only possible results from a measurement of a physical quantity are the *eigenvalues* $(\lambda_1, \ldots, \lambda_n)$ of the corresponding operator.
   6. Quantum mechanics is fundamentally non-deterministic. The outcome of a measurement is a probabalistic process. Upon making a measurement, the probability of obtaining λ_m as the result is $|a_m|^2$, where a_m is the expansion coefficient corresponding to the eigenstate e_m associated with eigenvalue λ_m. Note that this tells us that the proper normalization for *any* vector in the Hilbert space (including the basis eigenstates) is the one where the squares of the expansion coefficients sum to one. That is $\sum_j |a_j|^2 = 1$.
   7. Upon making a measurement, the wavefunction becomes equal to the eigenstate corresponding to the observed eigenvalue. This is known as *collapse of the wavefunction*.

   We begin by considering a system with only two possible states. One possible example is a simple model of a hydrogen atom where the single electron can be either in the ground or excited state. We'll use the energy of the atom as the physically-measurable quantity and work with the corresponding energy operator (which is known as the *Hamiltonian*, H). This operator has two eigenstates that correspond physically to the ground and excited states. We'll use the symbols e_g and e_e to represent these eigenstates and use E_g and E_e to represent the corresponding eigenvalues (these are the energies of these states).

   If we use (e_g, e_e) as an orthonormal basis for the Hilbert space, the operator H has the matrix representation $H = \begin{bmatrix} E_g & 0 \\ 0 & E_e \end{bmatrix}$. In this basis, the eigenstates themselves obviously have matrix representations $e_g = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_e = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   (a) Imagine that the atom is placed in a state where its wavefunction is given by $\psi = \sqrt{1/3}\, e_g + \sqrt{2/3}\, e_e$. That is, the atom is placed in a state that is neither the

ground state, nor the excited state, but instead a particular linear combination of the two. (This is called a *superposition* state, and is a very common occurrence in quantum mechanics).

i. Show that this wavefunction is properly normalized.

Solution: We have to show that $\sum_j |a_j|^2 = 1$. For this wavefunction that is: $|\sqrt{1/3}|^2 + |\sqrt{2/3}|^2$, which does indeed equal 1.

ii. If you were to make a measurement of the atom's energy, what is the probability that you would measure E_g? What about E_e?

Solution: The probability associated with each outcome is just the magnitude squared of the coefficient in the expansion. So $P_{E_g} = |\sqrt{1/3}|^2 = 1/3$. Similarly, $P_{E_e} = |\sqrt{2/3}|^2 = 2/3$.

iii. Now imagine that you prepare a large number of systems in this superposition state. If you were to measure the energy of each one, what would the *average* of all your measurements be? (This is called the *expectation value* of the measurement).

Solution: Since we know the possible outcomes, and we know their values, the expectation value is just a weighted average:

$$E_{\text{expect}} = 1/3E_g + 2/3E_e.$$

iv. Imagine that you take an atom prepared in the superposition state, measure its energy and find a value E_g. If you then immediately make a second measurement of the energy, what is the probability of finding the value E_e? (HINT: This measurement is *not* a projection. However, think about our class discussion of the projection operator P_u^2. Concepts mentioned in that discussion will help you here.)

Solution: In this case, $P_{E_e} = 0$. Immediately after a measurement, the system is in the eigenstate associated with that eigenvalue. Since we found E_g in the first measurement, the system is in e_g. If we think of this as a linear combination of e_g and e_e, the coefficient for e_e is 0. That means that the probability of observing a value corresponding to that eigenstate is also 0.

(b) Now imagine that some slight perturbation is made to the system (e.g. a small magnetic field is applied to the atom). This changes the operator H. Still using the eigenstates of the unperturbed system as a basis, the matrix representation of the new operator is $H_{\text{perturb}} = \begin{bmatrix} E_g & \alpha \\ \alpha & E_e \end{bmatrix}$.

i. The new Hamiltonian H_{perturb} has new eigenvalues (E_1 and E_2). Find them.

Solution: Using our general solution for a 2×2 problem, we find the eigen-

values of this new matrix are:

$$E_1 = \frac{(E_g + E_e) - \sqrt{4\alpha^2 + (E_e - E_g)^2}}{2},$$

which becomes E_g in the $\alpha = 0$ limit, and

$$E_2 = \frac{(E_g + E_e) + \sqrt{4\alpha^2 + (E_e - E_g)^2}}{2},$$

which becomes E_e in the $\alpha = 0$ limit.

ii. Find the corresponding new eigenstates (e_1 and e_2). Work in the original (e_g, e_e) basis. Your result should show that the new eigenstates are α-dependent linear combinations of the original ground and excited states. Explicitly write e_1 and e_2 in this form.

Solution: This is a bit messier, but still straightforward. Remember to normalize the eigenstates! We find:

$$e_1 = \left(\frac{-[(E_e - E_g) + \sqrt{4\alpha^2 + (E_e - E_g)^2}]}{\sqrt{8\alpha^2 + (E_e - E_g)[1 + \sqrt{4\alpha^2 + (E_e - E_g)^2}]}} \right) e_g + \left(\frac{2\alpha}{\sqrt{8\alpha^2 + (E_e - E_g)[1 + \sqrt{4\alpha^2 + (E_e - E_g)^2}]}} \right) e_e,$$

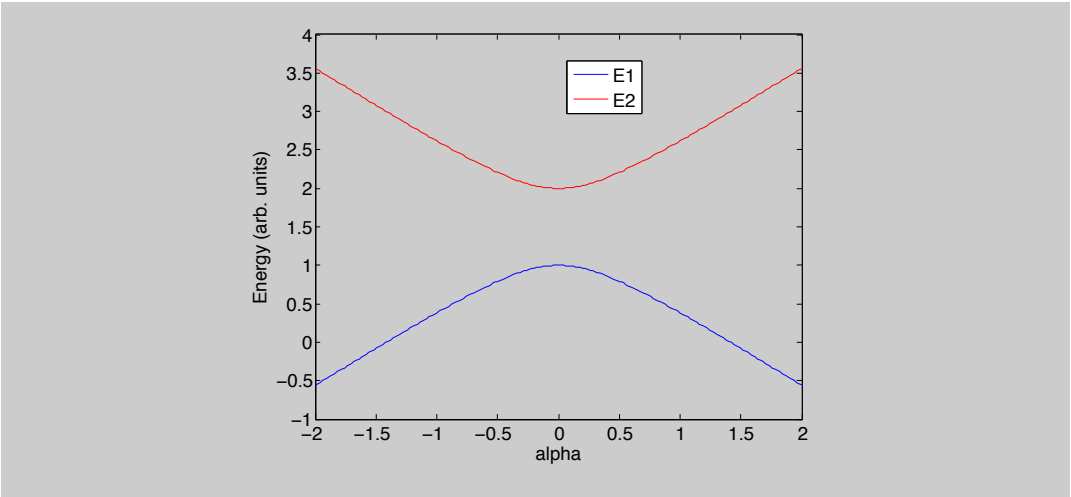
and

$$e_2 = \left(\frac{-[(E_e - E_g) - \sqrt{4\alpha^2 + (E_e - E_g)^2}]}{\sqrt{8\alpha^2 + (E_e - E_g)[1 - \sqrt{4\alpha^2 + (E_e - E_g)^2}]}} \right) e_g + \left(\frac{2\alpha}{\sqrt{8\alpha^2 + (E_e - E_g)[1 - \sqrt{4\alpha^2 + (E_e - E_g)^2}]}} \right) e_e.$$

Taking the $\alpha = 0$ limit is *very* tricky. The net result is that $e_1 \to e_g$ and $e_2 \to e_e$, so we do recover the unperturbed solution when we set the perturbation to zero.

iii. Take $E_g = 1$ and $E_e = 2$. Plot E_1 and E_2 as a function of $\alpha \in [-2, 2]$.

Solution:



NOTE: You're just a little too late. Had you formulated and worked part b of this homework problem in 1926, you might have won the Nobel Prize in physics.

3. Axler, Chapter 6, Exercise 10. Plot these functions on the domain $x \in [0, 1]$.

Solution: This is very straightforward. The first vector, 1, is already properly normalized. Thus $e_1 = 1$. To compute e_2, we begin by subtracting off the overlap with the first

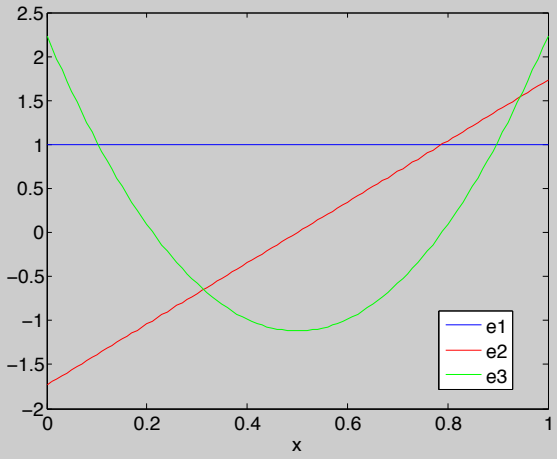
$$\begin{aligned} w_2 =& x - < x, e_1 > e_1 \\ =& x - < x, 1 > \\ =& x - \int_0^1 x \, dx \\ =& x - 1/2. \end{aligned}$$

To get e_2 we normalize w_2.

$$\begin{aligned} e_2 =& \frac{w_2}{||w_2||} \\ =& \frac{w_2}{\sqrt{< w_2, w_2 >}} \\ =& \frac{x - 1/2}{\sqrt{\int_0^1 (x - 1/2)^2 \, dx}} \\ =& 2\sqrt{3}(x - 1/2) \\ =& \sqrt{3}(2x - 1). \end{aligned}$$

A similar approach then finds $e_3 = \sqrt{5}(1 - 6x + 6x^2)$.

Plotting these functions gives:



4. Axler, Chapter 6, Exercise 21.

Solution: This is a minimization problem. First we need an orthogonal basis of our target space. Applying Gram-Schmidt to ((1,1,0,0),(1,1,1,2)) gives $e_1 = (1/\sqrt{2}, 1/\sqrt{2}, 0, 0)$ and $e_2 = (0, 0, 1/\sqrt{5}, 2/\sqrt{5})$.

To find the point in U that is closest to (1,2,3,4) (which is what the problem is effectively asking), we simply orthogonally project this point onto U.

$$v = < (1,2,3,4), e_1 > e_1 + < (1,2,3,4), e_2 > e_2.$$

With this definition, and the values of e_1 and e_2 above, we find $v = (3/2, 3/2, 11/5, 22/5)$.

5. Axler, Chapter 6, Exercise 25.

Solution: Note that the integral is a linear functional. It takes a vector $p(x)$ and returns a scalar. We saw in class that any linear functional can be represented as an inner product with a fixed vector. We use that approach here.

We define ϕ as our linear functional on $\mathcal{P}_2(\mathbb{R})$ by

$$\phi = \int_0^1 p(x)\cos(\pi x)\, dx.$$

We previously found an orthonormal basis for $\mathcal{P}_2(\mathbb{R})$ as: $(1, \sqrt{3}(2x-1), \sqrt{5}(1-6x+6x^2))$. We can represent ϕ as $< p, q >$ for all $p \in \mathcal{P}_2(\mathbb{R})$ with q defined as:

$$q = [\phi e_1]e_1 + [\phi e_2]e_2 + [\phi e_3]e_3.$$

Performing the integrals and simplifying leads us to the final result: $q(x) = \frac{12-24x}{\pi^2}$.

6. Axler, Chapter 6, Exercise 30. Remember that Axler uses * for the adjoint (we use †).

Solution: We begin by proving that T is injective i.f.f. $T^\dagger$ is surjective. We'll start with a statement that has the implication going both ways and then work with one side. This will prove both parts of the implication:

$$\begin{aligned} T \text{ is injective} &\Leftrightarrow \operatorname{null} T = 0 \\ &\Leftrightarrow (\operatorname{range} T^\dagger)^\perp = 0 \\ &\Leftrightarrow \operatorname{range} T^\dagger = W \\ &\Leftrightarrow T^\dagger \text{ is surjective.} \end{aligned}$$

The second part can then be proved immediately by exchanging T and $T^\dagger$ in the first proof.

7. Please estimate how much *productive* time you spent completing this assignment (watching television with the assignment in your lap does *not* count as productive time!).