Coupled Wave Theory for Thick Hologram Gratings

By HERWIG KOGELNICK

(Manuscript received May 23, 1969)

A coupled wave analysis is given of the Bragg diffraction of light by thick hologram gratings, which is analogous to Phareson's treatment of acoustic gratings and to the "dynamical" theory of X-ray diffraction. The theory remains valid for large diffraction efficiencies where the incident wave is strongly depleted. It is applied to transmission holograms and to reflection holograms. Spatial modulations of both the refractive index and the absorption constant are allowed for. The effects of loss in the gratings and of slanted fringes are also considered. Algebraic formulas and their numerical evaluations are given for the diffraction efficiencies and the angular and wavelength sensitivities of the various hologram types.

1. INTRODUCTION

Holographic recording in thick media ("volume recording") is of particular interest for high-capacity information storage, for color holography and for efficient white-light display of holograms. The high efficiency of light conversion which is attainable with thick dielectric holograms is also important for microimaging, and it may make it practical to use holographic optical components (for example, gratings) in a variety of optical systems.

In thick holograms it is light diffraction at or near the Bragg angle...
waves in thick holograms

Closely related to the diffraction in thick holograms are also the diffraction of electrons in lattices and the diffraction of X-rays in crystals. The dynamical theory of X-ray diffraction is also a theory of coupled waves and its application to holography has already been suggested.

We have earlier reported some of the results and an outline of the coupled wave theory for hologram gratings. Here we propose to give further results and a more detailed account of the basic assumptions and the analysis. We give analytic formulas for the various hologram types as well as numerical evaluations which include results on the angular sensitivities and the influence of loss and slit.

For simplicity the analysis is restricted to the holographic record of sinusoidal fringe patterns which we call hologram gratings. To some degree a more complicated hologram can be regarded as a multiplicity of such hologram gratings.

II. COUPLED WAVE ANALYSIS

2.1 Derivation of the Coupled Wave Equations

The coupled wave theory assumes monochromatic light incident on the hologram grating at or near the Bragg angle and polarized perpendicular to the plane of incidence. Only two significant light waves are assumed to be present in the grating: the incoming "reference" wave and the outgoing "signal" wave. Only these two waves obey the Bragg condition at least approximately, the other diffraction orders violate the Bragg condition strongly and are neglected. They should be of little influence on the energy interchange between S and R. The last assumption limits the validity of the coupled wave theory to thick hologram gratings. Section 6 gives a more detailed discussion of this limitation.

Figure 1 shows the model of a hologram grating which is used for our analysis. The z-axis is chosen perpendicular to the surface of the medium, the x-axis in the plane of incidence and parallel to the medium boundaries and the y-axis perpendicular to the paper. The fringe planes are oriented perpendicular to the plane of incidence and slanted with respect to the medium boundaries at an angle \( \phi \). The fringes are shown dotted. The grating vector \( \mathbf{K} \) is oriented perpendicular to the fringe planes and is of length \( K = 2a/\lambda \), where \( \lambda \) is the period of the grating. The same average dielectric constant is assumed for the region inside and outside the grating boundaries. The angle of incidence measured in the medium is \( \theta \).

A generalization to parallel polarization is given in the appendix.
Wave propagation in the grating is described by the scalar wave equation

$$\nabla^2 E + k^2 E = 0,$$

(1)

where \( E(x, z) \) is the complex amplitude of the \( y \)-component of the electric field, which is assumed to be independent of \( y \) and to oscillate with an angular frequency \( \omega \). The propagation constant \( k(x, z) \) is spatially modulated and related to the relative dielectric constant \( \varepsilon(z, z) \) and the conductivity \( \sigma(z, z) \) of the medium by

$$k^2 = \frac{\omega^2}{c^2} \varepsilon - j\frac{\omega}{c} \sigma$$

(2)

where \( c \) is the light velocity in free space and \( \mu \) is the permeability of the medium which we assume to be equal to that of free space. In our model the constants of the medium are independent of \( y \). The fringes of the hologram grating are represented by a spatial modulation of \( \varepsilon \) or \( \sigma 

e = \mu_0 + \varepsilon_1 \cos(Kx) 
\sigma = \sigma_0 + \sigma_1 \cos(Kx) 

(3)

where \( \varepsilon_1 \) and \( \sigma_1 \) are the amplitudes of the spatial modulation, \( \varepsilon_0 \) is the average dielectric constant and \( \sigma_0 \) the average conductivity. \( \epsilon \) and \( \sigma \) are assumed to be modulated in phase. To simplify the notation we have used the radius vector \( x \) and the grating vector \( K \)

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} , \quad K = \begin{bmatrix} \sin \phi \\ 0 \\ \cos \phi \end{bmatrix} , \quad K = 2\pi/\Lambda.$$

Equations (2) and (3) can be combined in the form

$$k^2 = \beta^2 - 2j\beta \sigma + \sqrt{\varepsilon} \varepsilon \varepsilon_0, \quad \beta = 2\pi(\varepsilon_0)^{1/2} / \lambda, \quad \sigma = \mu_0\sigma_0/2(\varepsilon_0)^{1/2} ,$$

(4)

and the coupling constant \( \kappa \) was defined as

$$\kappa = \frac{1}{4} \frac{\lambda}{\lambda} \frac{\varepsilon_0(\varepsilon_0)^{1/2} - \mu_0\sigma_0(\varepsilon_0)^{1/2}}.$$

(5)

This coupling constant describes the coupling between the reference wave \( R \) and the signal wave \( S \). It is the central parameter in the coupled wave theory. For \( \kappa = 0 \) there is no coupling between \( R \) and \( S \) and, therefore, there is no diffraction.

Optical media are usually characterized by their refractive index and their absorption constant. We also find it convenient to use these parameters if the following conditions are met

$$2\pi/\lambda \gg \sigma_1, \quad 2\pi/\lambda \gg \sigma_1, \quad n \gg n_1 ,$$

(7)

which is true in almost every practical case. Here \( n \) is the average refractive index, and \( n_1 \) and \( \sigma_1 \) are the amplitudes of the spatial modulation of the refractive index and the absorption constant, respectively [compare equation (3)]. \( \lambda \) is the wavelength in free space. Under the above conditions we can write with good accuracy

$$\beta = 2\pi/\lambda$$

(8)
and for the coupling constant
\[ \kappa = \frac{\pi n_i}{\lambda} - j\alpha_i/2. \]  
(9)

The spatial modulation indicated by \( n_i \) or \( \alpha_i \) forms a grating which couples the two waves \( R \) and \( S \) and leads to an exchange of energy between them. We describe these waves by complex amplitudes \( R(z) \) and \( S(z) \) which vary along \( z \) as a result of this energy interchange or because of an energy loss from absorption. The total electric field in the grating is the superposition of the two waves
\[ E = R(z)e^{j\phi(z)} + S(z)e^{j\theta(z)}. \]  
(10)

The propagation vectors \( \phi \) and \( \theta \) contain the information about the propagation constants and the directions of propagation of \( R \) and \( S \). \( \phi \) is assumed to be equal to the propagation vector of the free reference wave in the absence of coupling. \( \theta \) is forced by the grating and related to \( \phi \) and the grating vector by
\[ d = \phi - \kappa \]  
(11)

which has the appearance of a conservation of momentum equation. \( \phi \) and \( \theta \) have been chosen to conform as closely as possible to our picture of the physical process of the diffraction in the grating. If the actual phase velocities differ somewhat from the assumed values, then these differences will appear in the complex amplitudes \( R(z) \) and \( S(z) \) as a result of the theory.

Figure 2 shows the vectors of interest and their orientation. The components of \( \phi \) are \( \phi_r \) and \( \phi_\theta \), which are given by
\[ \phi = \begin{bmatrix} \phi_r \\ \phi_\theta \end{bmatrix} = \begin{bmatrix} \sin \theta \\ \cos \theta \end{bmatrix}. \]  
(12)

From this and equation (11) follow the \( \phi \)-components \( \sigma_\alpha \) and \( \sigma_\beta \),
\[ d = \begin{bmatrix} \sigma_\alpha \\ \sigma_\beta \end{bmatrix} = \begin{bmatrix} \sin \theta - \frac{K}{\beta} \sin \phi \\ \cos \theta - \frac{K}{\beta} \cos \phi \end{bmatrix}. \]  
(13)

The vector relation (11) is shown in Fig. 3 together with a circle of radius \( \beta \). The general case is shown in Fig. 3a, where the Bragg condition is not met and the length of \( d \) differs from \( \beta \). Figure 2b shows the same diagram for incidence at the Bragg angle \( \theta_B \). In this special case the lengths of both, \( \phi \) and \( \theta \) are equal to the free propagation constant \( \beta \), and the Bragg condition
\[ \cos (\phi - \theta) = \frac{K}{2\beta} = \frac{\pi}{\lambda} \]  
(14)

is obeyed.

For a fixed wavelength the Bragg condition is violated by angular
deviations $\Delta \theta$ from the Bragg angle $\theta_0$. For a fixed angle of incidence a similar violation takes place for changes $\Delta \lambda$ from the correct wavelength $\lambda_0$. We write

$$\theta = \theta_0 + \Delta \theta,$$

and

$$\lambda = \lambda_0 + \Delta \lambda,$$

and assume in the following that the deviations $\Delta \theta$ and $\Delta \lambda$ are small.

Angular changes $\Delta \theta$ have very similar effects on the behavior of the grating as wavelength changes $\Delta \lambda$, and there is a close relation between the angular sensitivity and the wavelength sensitivity of thick hologram gratings. We get an idea of this relationship by differentiating the Bragg condition (14), from which results

$$\frac{d\theta}{d\lambda} = K/4\pi n \sin (\phi - \theta).$$

The $\theta - \lambda$ connection shows up in the dephasing measure $\vartheta$ which appears in the coupled wave equations and which is defined by

$$\vartheta = (\theta^2 - \sigma^2)/2\beta = K \cos (\phi - \theta) - K^2/4\pi n \lambda.$$

and which has been expressed in this form using equation (13). A Taylor series expansion of equation (17) yields the following expression for $\vartheta$ which is correct to the first order in the deviations $\Delta \theta$ and $\Delta \lambda$:

$$\vartheta = \Delta \theta \cdot K \sin (\phi - \theta) - \Delta \lambda \cdot K^2/4\pi n.$$

Note that the deviations $\Delta \theta$ and $\Delta \lambda$ which produce equal dephasing $\vartheta$ are related by equation (16).

We are now ready to derive the coupled wave equations. We combine equations (1) and (4), and insert the expressions of (10) and (11). Then we compare the terms with equal exponentials ($e^{-i\sigma z}$ and $e^{-i\vartheta z}$) and arrive at

$$R'' - 2jR'\rho - 2j\vartheta R + 2\sigma S = 0$$

and

$$S'' - 2jS'\sigma - 2j\vartheta S + (\theta^2 - \sigma^2)S + 2\sigma R = 0,$$

where the primes indicate differentiation with respect to $z$. The waves generated in the directions of $\sigma + K$ and $\sigma - K$ are neglected, together with all other higher diffraction orders. In addition we assume that the energy interchange between $S$ and $R$ is slow and that energy is absorbed slowly, if at all. This allows us to neglect $R''$ and $S''$. We will check the results of the theory later for a more detailed justification of this last step. We can now introduce equation (18) and rewrite the above equations in the form

$$c_\sigma R + \alpha R = -j\vartheta S$$

$$c_\sigma S + (\alpha + \beta)S = -j\sigma R.$$

These are the coupled wave equations which are the basis for our analysis. The abbreviations $c_\sigma$ and $c_\vartheta$ stand for the expressions

$$c_\sigma = \rho/\beta = \cos \theta$$

$$c_\vartheta = \vartheta/\beta = \cos \theta - K/\beta \cos \phi.$$

Our physical picture of the diffraction process is reflected in the coupled wave equations. A wave changes in amplitude along $z$ because of coupling to the other wave $c_\sigma S$ or absorption $c_\sigma R$. For deviations from the Bragg condition $S$ is forced out of synchronism with $R$ and the interaction decreases $\vartheta S$.

The energy balance of the coupled-wave model is described by the relation

$$(c_\sigma RR^* + c_\vartheta SS^*)' + 2\alpha (RR^* + SS^*) + j(\sigma - \epsilon^2)(RS^* + R^*S) = 0$$

where the asterisk denotes a complex conjugate. This is easily derived from equations (21) and (22) by multiplying them with $R^*$ and $S^*$, respectively, and adding the results together with the complex conjugate results. The presence of the obliquity factors $c_\sigma$ and $c_\vartheta$ in the first part of equation (24) indicates that it is the power flow of the two waves in the $z$ direction that enters the energy balance. In the absence of ohmic loss this power flow is conserved. The second and the third part in the equation describe the energy loss resulting from absorption in the grating. They correspond to the relevant terms of $\sigma EE^*.$

2.2 Solution of the Coupled Wave Equations

It is straightforward to obtain the general solution of the coupled wave equations, which is

$$R(z) = r_1 \exp (\gamma z) + r_2 \exp (\gamma z)$$

$$S(z) = s_1 \exp (\gamma z) + s_2 \exp (\gamma z)$$

where $\gamma$, $\sigma$, and $\vartheta$ are the roots of the characteristic equation.
where the \( r_i \) and \( s_i \) are constants which depend on the boundary conditions. To determine the constants \( r_i \) we insert equations (25) and (26) into the coupled wave equations and obtain

\[
(c_s \gamma_i + \alpha) r_i = -j \varepsilon, \quad i = 1, 2.
\]

(27)

\[
(c_s \gamma_i + \alpha + j \beta) s_i = -j \varepsilon_r, \quad i = 1, 2.
\]

(28)

After multiplying the equations with each other we get a quadratic equation for \( \gamma_i \),

\[
(c_s \gamma_i + \alpha)(c_s \gamma_i + \alpha + j \beta) = -\varepsilon_i^2,
\]

(29)

with the solution

\[
\gamma_{1,2} = \frac{1}{2} \left( \frac{\alpha}{c_s} + \frac{\alpha + j \beta}{c_s} \pm \frac{1}{2} \left[ \left( \frac{\alpha}{c_s} - \frac{\alpha + j \beta}{c_s} \right)^2 - 4 \frac{\varepsilon_i^2}{c_s^2} \right]^{1/2} \right).
\]

(30)

At this point we divert briefly from the main derivation, because now we have the means to check the validity of neglecting \( R' \) and \( S'' \) in Section 2.1. This step is justified if the conditions \( R' \ll \varepsilon, R'' \) and \( S'' \ll \varepsilon, S' \) are obeyed. In view of equations (25) and (26) this will happen if \( \gamma_i \ll \beta \). According to equation (30) the above requirement is met if \( \Delta \beta \ll 1 \) and if the inequalities of equation (7) are satisfied (which is usually the case).

Continuing the coupled wave analysis, we have to determine the constants \( r_i \) and \( s_i \). To do this we have to introduce boundary conditions into our model. These are different for transmission holograms and for reflection holograms. Figure 4 gives an indication of this. For both hologram types the reference wave \( R \) is assumed to start with unit amplitude at \( z = 0 \). It decays as it propagates to the right and couples energy into \( S \). In transmission holograms the signal \( S \) starts out with zero amplitude at \( z = 0 \) and propagates to the right \( (c_s > 0) \). In reflection holograms the signal travels to the left \( (c_s < 0) \) and it starts with zero amplitude at \( z = d \).

Let us first analyze transmission holograms where \( c_s > 0 \). Here, the boundary conditions are

\[
R(0) = 1, \quad S(0) = 0
\]

(31)
as discussed before. If we insert these boundary conditions into equations (25) and (26), it follows immediately that

\[
r_1 + r_2 = 1,
\]

(32)

and

\[
s_1 + s_2 = 0.
\]

(33)

Combining these relations with equation (28) we obtain

\[
s_1 = -s_2 = -j \varepsilon / c_s (\gamma_1 - \gamma_2).
\]

(34)

Introducing these constants in equation (26) we arrive at an expression for the amplitude of the signal wave at the output of the grating

\[
S(d) = \frac{c}{c_s (\gamma_1 - \gamma_2)} (\exp(\gamma_2 d) - \exp(\gamma_1 d)).
\]

(35)

This is a general expression, which is valid for all types of thick transmission holograms including the cases of off-Bragg incidence, of lossy gratings and of slanted fringe planes.

The analysis of reflection holograms follows a pattern similar to the above. We have \( c_s < 0 \) and boundary conditions given by

\[
R(0) = 1, \quad S(d) = 0.
\]

(36)

Fig. 4—Wave propagation in (a) transmission and (b) reflection holograms. The reference wave \( R \) decays while it propagates to the right. In (a) the signal \( S \) travels to the right and gains with \( z \), while in (b) \( S \) travels to the left and gains with decreasing \( z \). The shading indicates the orientation of the fringes.
The output plane for the signal wave is, now, at $z = 0$, and $S(0)$ is the output amplitude of interest. Inserting the boundary conditions in equations (25) and (26) yields

$$r_1 + r_2 = 1$$

and

$$s_1 \exp (\gamma_d d) + s_2 \exp (\gamma_d d) = 0. \quad (36)$$

To proceed with our derivation we rewrite the above relation for $s_1$ and $s_2$ in the form

$$s_1 (\exp (\gamma_d d) - \exp (\gamma_d d)) = (s_1 + s_2) \exp (\gamma_d d)$$

$$s_2 (\exp (\gamma_d d) - \exp (\gamma_d d)) = -(s_1 + s_2) \exp (\gamma_d d). \quad (37)$$

Then we sum equation (28) for $i = 1$ and $i = 2$ and obtain the relation

$$-j \pi (r_1 + r_2) = -j \pi = (s_1 + s_2)(\alpha + j \theta) + c_s (\gamma_1 s_1 + \gamma_2 s_2). \quad (38)$$

Using the relations (37) to substitute the sum $(s_1 + s_2)$ for the terms $s_1$ and $s_2$ in this equation we finally arrive at the result for the amplitude $S(0)$ of the output signal of a reflection hologram

$$S(0) = s_1 + s_2 = -j \pi \left( \frac{\alpha + j \theta + c_s \gamma_1 \exp (\gamma_d d) - \gamma_2 \exp (\gamma_d d)}{\exp (\gamma_d d) - \exp (\gamma_d d)} \right). \quad (39)$$

This is, again, a formula of quite general validity, including off-Bragg incidence, loss, and slant.

In the following sections we discuss the behavior of transmission and reflection holograms in greater detail, using the general formulas derived above. In these discussions a parameter of prime interest is the diffraction efficiency $\eta$, which is defined as

$$\eta = \frac{|c_s|}{c_s} S S^* \quad (40)$$

where $S$ is the (complex) amplitude of the output signal for a reference wave $R$ incident with unit amplitude. $\eta$ is the fraction of the incident light power which is diffracted into the signal wave. $S$ is equal to $S(d)$ for transmission holograms and equal to $S(0)$ for reflection holograms in the notation of this section. But for reasons of simplicity we omit the arguments in the following sections. The obliquity factors $c_s$ and $c_d$ appear in the above definition for the same reason they have appeared in the energy balance of equation (24): in the absence of loss it is the power flow in the $z$ direction which is conserved.

For slanted gratings another important parameter is the slant factor $c_s$ which is defined as the ratio between the obliquity factors

$$c_s = c_{ds}/c_d = -\cos \theta/\cos (\theta_0 - 2\delta)$$

which we have expressed here, for Bragg incidence, in terms of the angle of incidence $\theta$, and the slant angle $\delta$. Figure 5 indicates lines of constant $c_s$ as a function of $\theta$, and for reflection holograms $c_s$ is negative ($c < 0$). In the diagram transmission and reflection holograms are separated by the line for $c = \infty$.

III. TRANSMISSION HOLOGRAMS

In this section we discuss transmission holograms in greater detail. We give algebraic formulas and their numerical evaluations for the diffraction efficiencies and the angular and wavelength sensitivities of dielectric and of absorption gratings. This includes results on the influence of loss and slant.
It is convenient to write the various diffraction formulae in terms of parameters \( \nu \) and \( \xi \), which are redefined for each grating type. In these parameters are lumped together the constants of the medium \((n, \alpha, n_1, \alpha_1, \kappa)\), the obliquity factors \((c_n, c_\alpha)\), the wavelength, the grating thickness \(d\), and the dephasing measure \(\theta\). By using \( \nu \) and \( \xi \) various trade-offs become immediately apparent.

We recall that, for transmission holograms, \(c_n\) is positive and the output signal appears at \(z = d\). Combining equations (30) and (34) we obtain a general formula for the signal amplitude \(S\) of a transmission grating

\[
S = -i \left( \frac{c_\alpha}{c_n} \right)^{1/2} \exp \left( -ad/c_\alpha \right) e^{-i \cdot \sin \left[ \left( \nu^2 - \xi^2 \right) \right]} / \left[ 1 - \nu^2 / \nu^2 \right],
\]

\[
\nu = ad/c_\alpha \nu_0 \sin \theta,
\]

\[
\xi = \frac{1}{2} \left( \frac{\alpha}{c_n} \frac{\alpha_1}{c_\alpha} - \frac{\theta}{c_n} \right),
\]

where \(\alpha\) is the coupling constant given in equation (9), \(\theta\) the dephasing measure of equation (18), \(c_n\) and \(c_\alpha\) are the obliquity factors of equation (22), \(\alpha\) is the absorption constant and \(d\) the grating thickness.

In the above formula the parameters \(\nu\) and \(\xi\) are, in general, of complex value.

### 3.1 Lossless Dielectric Gratings

For completeness we give the formulas for the lossless dielectric grating. For the unslanted case of this grating these formulas have been previously obtained by several workers whose prime interest was light diffraction by acoustic waves.\(^{20,21,22}\) For this grating type it is easy to include the effect of slanted fringes.\(^*\) For the lossless dielectric grating we have a coupling constant \(\kappa = \pi n_1 / \lambda\) and \(\alpha = 0\). Equation (41) can be rewritten in the form

\[
S = -i \left( \frac{c_\alpha}{c_n} \right)^{1/2} e^{-i \sin \left( \nu^2 + \xi^2 \right)/\left(1 + \xi^2 / \nu^2 \right)} / \left(1 - \nu^2 / \nu^2 \right),
\]

\[
\nu = \pi n_1 / \lambda (c_\alpha c_n),
\]

\[
\xi = \Delta \theta / 2c_\alpha,
\]

where \(\nu\) and \(\xi\) have been redefined and are real-valued. The associated formula for the diffraction efficiency is

\* Slant was also included in the treatment of dielectric gratings in Ref. 29.

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For significant deviations from the Bragg condition the parameters \(\nu\) and \(\xi\) are of equal order of magnitude, and we can take \(\nu\) as independent of \(\Delta\theta\) or \(\Delta\lambda\) without causing an appreciable change in the predictions of equation (43). In this equation the angular and wavelength deviations are represented by the parameter \(\xi\) which can be written in the form

\[
\xi = \Delta \theta / K d \sin \left( \phi - \phi_0 \right) / 2c_\alpha
\]

\[
= -\Delta \lambda / K d / \pi n_1 \nu_0
\]

by using equation (18).

The angular and wavelength sensitivities of lossless dielectric gratings are shown in Fig. 6, where the efficiencies as given by equation (43) are plotted (normalized) as a function of \(\xi\) for three values of \(\nu\). The figure shows the sensitivity of gratings with \(\nu = \pi / 4\) and a peak diffraction efficiency of \(\eta_\nu = 0.5\), with \(\nu = \pi / 2\) and a peak efficiency of \(\eta_\nu = 1\), and with \(\nu = 3\pi / 4\) and \(\eta_\nu = 0.5\). We notice that the half-power points are reached for values of \(\xi = 1.5\). There is some narrowing in the sensitivity curves for increasing values of \(\nu\) and a marked increase in the side lobe intensity.

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**Fig. 6.** Transmission holograms—the angular and wavelength sensitivity of lossless dielectric gratings with the normalized efficiencies \(\eta_\nu / n_1\) as a function of \(\xi\).
for an additional exponential term containing the absorption constant $\alpha$. This term decreases the peak efficiency and it changes the angular sensitivity of the grating. But this change is very small, even for high loss values, as illustrated in Fig. 7. This figure compares the angular sensitivities of a lossless grating ($D_a = 0$) with that of a grating of high loss ($D_a = 2$) for a parameter value of $\nu = \pi/2$, a Bragg angle of $\theta_0 = 30^\circ$, and an optical grating thickness of $\beta d = 2 \nu \pi d / \lambda = 50$.

The loss parameter $D_a$ was defined as

$$D_a = \alpha d / \cos \theta_0$$

which is closely related to the conventional photographic density $D$ (except that $D_a$ is measured in the direction of the reference wave given by $\theta_0$). A value of $D_a = 2$, which is the parameter used for the dashed curve, represents very high loss, with a decrease of the peak efficiency by a factor of about 50. Still, the differences of the two sensitivity curves are very small and consist mostly of an angular shift. The differences are even smaller for larger losses of $\beta d$ (we checked up to $\beta d = 200$, and, of course, for smaller values of $D_a$). The main conclusion is that the presence of loss has very little influence on the angular sensitivity of a dielectric transmission grating. This is probably because absorption influences the phase relations between the waves $R$ and $S$ very little. It agrees with observations by Belvaux.31

Next let us consider the influence of loss on the efficiency of a slanted dielectric grating. For simplicity we assume Bragg incidence, that is, $\beta = 0$. The obliquity factors are positive and given by $c_s = \cos \theta_0$ and $c_p = -\cos (\theta_0 - 2\phi)$. For this case we can write equation (41) for the signal amplitude $S$ in the form

$$S = \frac{-\xi c_p}{c_s} \exp \left[ -\frac{1}{2} D_a (1 + c) \sin (\xi^2 - c^2) \right]$$

$$\xi = \frac{\nu \pi d}{\lambda (c_s c_p)}$$

$$\nu = \frac{\xi d}{2 \cos \theta} = \Delta \theta - c \sin \theta_0$$

where we have used the loss parameter $D_a$ as above in equation (47), and the slant factor $c$

$$D_a = \alpha d / \cos \theta_0$$

$$c = c_s / c_p = -\cos (\theta_0 - 2\phi)$$

Figure 8 shows the diffraction efficiency of slanted gratings as calculated from equation (48). The efficiencies are plotted as a function
of the slant factor $c$ for various values of $D_s$, and for a value of $v = \pi/2$ which corresponds to the maximum attainable efficiencies. Similar curves for $v = \pi/4$ and $v = 3\pi/4$ and the same $D_s$ values are almost identical to the curves of Fig. 8, except that the efficiency scale is reduced to a maximum efficiency of 0.5. This implies that for the range of chosen parameter values the exponential factor in equation (45) dominates in predicting the slant-dependence of the diffraction efficiency.

The results show that, for higher efficiencies, the grating prefers small $v$-values, assuming constant $\theta_b$ and $D_s$. This is a preference of small exit angles for $S$ which means that we get the best efficiency if the signal wave leaves the grating on the shortest possible path it has been generated.

### 3.3 Unslanted Absorption Gratings

When one records holograms in conventional photographic emulsions one produces absorption gratings (bleaching can convert this into a dielectric grating). In an absorption grating there is no spatial modulation of the refractive index ($n_r = 0$) and the coupling is provided by a modulation ($n_i$) of the absorption constant. We have, then, an imaginary coupling constant $\kappa = -jn_i/2$. In this section we study the efficiencies and the angular and wavelength sensitivities of unslanted absorption gratings where $\phi = \pi/2$ and $c_s = c_r = \cos \theta$. From equation (41) we obtain for the signal amplitude

$$S = -\exp (-ad/c_r) \cdot e^{-i\phi} \cdot \sin (v - \xi)/0.5 \cdot (1 - e^{-v})$$

$$v = \alpha_d/2 \cos \theta$$

$$\xi = d\phi/2 \cos \theta \approx \Delta \theta - \Delta d - \sin \theta_b = -\frac{1}{2} (\Delta \phi/\lambda) \cdot Kd \tan \theta_b$$

where $v$ and $\xi$ are real-valued, and equation (18) was used to express the parameter $\xi$ again in various forms, showing explicitly the angular deviations $\Delta \theta$ and the wavelength deviation $\Delta \phi$ from the Bragg condition.

For Bragg incidence we have $\xi = 0$ and obtain from the above a formula for the diffraction efficiency $\eta$ of absorption gratings

$$\eta = \exp (-2ad/c_r \cos \theta_b) \cdot \sin^2 (\alpha_d/2 \cos \theta_b).$$

(50)

As we exclude the presence of negative absorption (gain) in the medium, there is an upper limit for the amplitude $a_0$ of the assumed sinusoidal modulation, which is $a_0 = a$. The highest diffraction efficiency possible for an absorption grating is reached in the limiting case $a_0 = a$ for a value of $ad/c_r \cos \theta_b = 1.3$. According to equation (50) this maximum efficiency has a value of $\eta_{max} = 12/27$, or 3.7 percent.

Figure 9 shows values for the diffraction amplitude $S$ of absorption gratings as computed from equation (50) as a function of the modulation amplitude $a_0$ and for various values of the depth of modulation. For convenience we have again used less parameters, which are $D_b = ad/c_r \cos \theta_b$ and $D_x = a_0 d/c_r \cos \theta_b$. $D_x$ is a measure for the amplitude of the spatial modulation and $D_b/D_x = a_0/a_0$ indicates the modulation depth. The dashed curves for constant $D_b$ show the grating behavior for constant background absorption. We have plotted $S$ on a linear scale in order to identify the regions of linear grating response. Note that a good linear response and relatively good efficiency is obtained if the absorption background is held constant to a value of about $D_b = 1$.

Equation (49) predicts also the angular sensitivity and the frequency sensitivity of absorption gratings. Such sensitivity curves are plotted in Fig. 10 for the special case of $a_0 = a$ and values of $v = D_b/2 = 1$ (dashed) and $v = \pi/2$ in $3 = 0.55$. For the latter parameter value the peak efficiency of 3.7 percent is reached, and for $v = 1$ we have a peak efficiency of 2.5 percent. In the figure the relative efficiencies are plotted as functions of the parameter $v$ $\xi$. We note that there is very little difference between the sensitivity curves for the two $v$-values chosen. We have also computed the sensitivity for smaller values of $v$ (0.2, 0.4), but the resulting curves differ so little from the ones shown that we have omitted them from the figure. The sensitivity curves are very