FIBER ANALYSIS

- Develop Field values and propagation rules
  a. Propagation and attenuation constants
  b. Bessel Function solutions

- Core and cladding solutions – boundary value solution
  ABCD Matrix for E&M-fields in cylindrical coordinates

- Characteristic Equation for transverse E&M fields

- TE, TM Modes and simplified characteristic equation $\nu = 0$

- Hybrid modes $\nu \neq 0$

- LP Modes – weakly guiding condition with $n_1 \approx n_2$; degeneracy of modes

- Mode cut off with $V$; Roots of the Bessel functions

- Mode Profile of lowest order mode – similar to a Gaussian; Power in the fiber core and the cladding.
Homogeneous Wave Equation

\[ \nabla^2 E - \mu E \frac{\partial^2 E}{\partial z^2} = 0 \]

\[ E = \varepsilon E_0 \quad \mu = \mu_0 \quad c = \frac{1}{\sqrt{\varepsilon \mu}} \]

\[ |k| = \sqrt{\varepsilon \mu} \omega = \frac{\omega}{c_0} = k_0 m \]

\[ \nabla^2 E + k_0^2 m^2 E = 0 \]

In cylindrical coordinates:

\[ E(r, \theta, z) = R E_r (r, \theta, z) + \phi E_\theta (r, \theta, z) + z E_z (r, \theta, z) \]

(radial component) \quad (azimuthal component) \quad (z-component)

(View Graph of Field propagation) \quad r \rightarrow \infty

r and \( \theta \) components can change their form

E_z component does not become ambiguous with other field components.

\[ \text{Solving for the } E_z \text{ field component provides} \]

a unique solution.

Scalar Wave Equation in Cylindrical Coordinates:

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} + \frac{\partial^2 E_z}{\partial z^2} + (k_0 m)^2 E_z = 0 \]

Once values for \( E_z \) are found, Maxwell's Equations can be used to obtain \( E_r \) and \( E_\theta \).
Circular Waveguides:

- The modes in a circular waveguide or fiber are found in a similar manner.
- The eigenvalues of a fiber are limited to the range:
  \[ k_{o}n_1 > \beta > k_{o}n_2. \]
- The characteristic equations for fibers are transcendental Bessel functions instead of sines or cosines.
- There are two mode numbers that are used to specify the mode: \( m \) is the radial mode number and \( \nu \) is the angular mode number.
- The radial mode number is associated with the radial component of the field and the angular mode number with the angular field component.
- A skew ray propagating through a fiber shows these characteristics.
Cylindrical Waveguides:

Assumed solution for $E_z$:

$$E_z(r, \phi, z) = R(r)\Phi(\phi)Z(z)$$

After substituting into scalar wave equation in cylindrical coordinates:

$$r^2 R'' \Phi Z + \frac{1}{r} R' \Phi Z + \frac{1}{r^2} R \Phi' Z + R \Phi Z' + k_o n^2 R \Phi Z = 0$$

Multiplying by $\frac{r^2}{R \Phi Z}$:

$$r^2 \frac{R''}{R} + r \frac{R'}{R} + \frac{\Phi''}{\Phi} + r^2 \frac{Z'}{Z} + (k_o n r)^2 = 0$$

Due to the predominant propagation of the field along the $z$ axis an oscillatory characteristic is assumed for the $z$ dependence.

$$Z(z) = e^{-j \beta z}$$

where $\beta$ is the $z$ component of the propagation vector $k$ within the waveguide.

After differentiation

$$\frac{Z'}{Z} = -\beta^2$$

This value can be substituted back into the differential wave equation

$$r^2 \frac{R''}{R} + r \frac{R'}{R} + \frac{\Phi''}{\Phi} - r^2 \beta^2 + (k_o n r)^2 = 0$$
or
\[
 r^2 \frac{\dot{R}}{R} + r \frac{\dot{R}}{R} - r^2 \beta^2 + (k_o n r)^2 = -\frac{\Phi^*}{\Phi} = \nu^2
\]

This expression shows explicitly that the \( r \) and \( \varphi \) dependencies can be separated. It also allows a solution for the \( \Phi \) functional component:

\[
\Phi^*(\varphi) = -\nu^2 \Phi
\]

with solutions of the form

\[
\Phi(\varphi) = A e^{i\nu \varphi} + c.c.
\]

Circular symmetry within the waveguide requires

\[
\Phi(\varphi) = \Phi(\varphi + 2\pi)
\]

therefore \( \nu \) must be an integer since

\[
\exp\left\{ j\nu(\varphi + 2\pi) \right\} = \exp\left\{ j(\nu \varphi + \nu 2\pi) \right\} = \exp\left\{ j\nu \varphi \right\}
\]

Another way to think of this is that when going around the cross section of the fiber at a constant radius the field distribution repeats itself.

The differential equation can now be written solely as a function of \( r \):

\[
r^2 \frac{\partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} + r^2 \left( k_o^2 n^2 - \beta^2 - \frac{\nu^2}{r^2} \right) R = 0.
\]

This differential equation is in the form that has Bessel function solutions. Bessel functions are transcendental equations with properties that are similar to sines and cosines. The sign of the argument in \( (\ ) \) determines the type of Bessel function solution to the differential equation just as it did for the planar waveguide case.
CASE 1:

\[ \left( k_o^2 n^2 - \beta^2 - \frac{\nu^2}{r^2} \right) > 0 \]

The solution to the differential equation when this factor is positive are Bessel functions of the first kind of order \( \nu \): \( J_\nu(\kappa r) \) where \( \nu \) is an integer.

As before:

\[ \kappa^2 = k_o^2 n^2 - \beta^2 \]

CASE 2:

\[ \left( k_o^2 n^2 - \beta^2 - \frac{\nu^2}{r^2} \right) < 0 \]

In this case solutions are modified Bessel functions of the second kind order \( \nu \); \( K_\nu(\gamma r) \) where

\[ \gamma^2 = \beta^2 - k_o^2 n^2 \]

and corresponds to the decay constant in the cladding of the fiber.

**Basic Properties:**
(viewgraphs)

\( J_\nu(\kappa r) \) \( \sim \) periodic along the radial axis of the fiber

\( J_0(\kappa r) \) \( \sim \) the only Bessel function with a non-zero value at \( r = 0 \). Other \( J_{\nu\neq0}(\kappa r) \) functions are zero at \( r = 0 \).

For large values of \( (\kappa r) \) the functions \( J_\nu(\kappa r) \) can be approximated as
Graphs show the first three Bessel functions of the first kind, $J_0(x)$, $J_1(x)$, and $J_2(x)$, and of the second kind, $K_0(x)$, $K_1(x)$, and $K_2(x)$. 
\[ J_v (\kappa r) = \sqrt{\frac{2}{\pi \kappa r}} \cos (\kappa r - \nu \pi / 2 - \pi / 4) \]

the function appears as a damped sine wave.

The \( J_v \) Bessel functions describe the radial standing waves within the core of the fiber.

\( K_v (\gamma r) \) describes the field distribution in the radial direction within the cladding. It has a monotonically decreasing characteristic. Higher orders decrease at a slower rate.

Approximate form of \( K_v (\gamma r) \) with large values of \( \gamma r \):

\[ K_v (\gamma r) = \frac{e^{-\gamma r}}{\sqrt{2\pi \gamma r}} \]

This implies that at large values of \( r \) all orders of \( K_v (\gamma r) \) look similar.

The factor \( \frac{1}{\sqrt{2\pi \gamma r}} \) is the natural decrease of a wave amplitude as it expands with radius. The exponential factor represents evanescent decay.

---

Field within a Step Index (SI) Fiber:
\( \beta \) must satisfy the conditions:
\[
k_o n_2 < \beta < k_o n_1
\]

- In the cladding the field amplitude decays \( \therefore \) choose \( K_\nu(\gamma r) \) as the solution for \( r > a \).

- In the core the field oscillates \( \therefore \) choose \( J_\nu(\kappa r) \) for \( r < a \).

Using these results the field solutions for \( E_z \) and \( H_z \) are of the form:

\[ r \leq a:\]
\[
E_z(r, \varphi, z) = AJ_\nu(\kappa r)e^{jv \varphi}e^{-j\beta z} + c.c.
\]
\[
H_z(r, \varphi, z) = BJ_\nu(\kappa r)e^{jv \varphi}e^{-j\beta z} + c.c.
\]

and

\[ r > a:\]
\[
E_z(r, \varphi, z) = CK_\nu(\chi r)e^{jv \varphi}e^{-j\beta z} + c.c.
\]
\[
H_z(r, \varphi, z) = DK_\nu(\chi r)e^{jv \varphi}e^{-j\beta z} + c.c.
\]

\( v \) specifies a mode number;

A, B, C, and D are constants determined from the boundary conditions.
Once \( E_z \) is determined for the core and cladding regions \( E_r, E_\phi, H_r, \) and \( H_\phi \) can be found using Maxwell's Eq. 1.

From \( \nabla \times H = \frac{\partial D}{\partial t} = j \omega E \)
\[ \nabla \times E = -\frac{\partial B}{\partial t} = -N \frac{\partial H}{\partial t} = -N j \omega \nabla \times H \]

Expanding the \( \nabla \times \) terms in cylindrical components and collecting terms

\[ E_\phi = -\frac{i}{\alpha^2} \left[ \frac{\beta}{r} \frac{\partial E_\phi}{\partial \phi} - \omega N \frac{\partial H_\phi}{\partial r} \right] \]
\[ E_r = -\frac{i}{\alpha^2} \left[ \frac{\mu \omega}{r} \frac{\partial H_\phi}{\partial \phi} + \beta \frac{\partial E_\phi}{\partial r} \right] \]
\[ H_d = -\frac{i}{\alpha^2} \left[ \omega \varepsilon \frac{\partial E_\phi}{\partial \phi} + \frac{\beta}{r} \frac{\partial H_\phi}{\partial \phi} \right] \]
\[ H_r = -\frac{i}{\alpha^2} \left[ \beta \frac{\partial H_\phi}{\partial r} - \frac{\omega \varepsilon}{r} \frac{\partial E_\phi}{\partial \phi} \right] \]

with \( \alpha^2 = \mu_0 n_i^2 - \beta^2 > 0 \) in the core region
\( \lessgtr 0 \) in the cladding region

We still need to determine the relations between \( A, B, C, + D \) and the allowed values for \( K \) and \( \delta \).
Continuity of \( E_\perp \) at \( r = a \):

\[
A J_\nu(\lambda a) = C K_\nu(\delta a)
\]

\[
\therefore \quad C = \frac{J_\nu(\lambda a)}{K_\nu(\delta a)} A
\]

From the continuity of \( H_\perp \) at \( r = a \):

\[
D = \frac{J_\nu(\lambda a)}{K_\nu(\delta a)} B
\]

The other coefficients with continuous components at the \( r = a \) interface are \( E_\phi \) and \( H_\phi \).

From the continuity of \( E_\phi \) at \( r = a \):

\[
y_B = \frac{j \omega \mu_0}{\mu_0} \left[ \frac{1}{\lambda^2} + \frac{1}{\delta^2} \right] \left[ \frac{J_\nu'(\lambda a)}{K_\nu'(\delta a)} + \frac{K_\nu(\delta a)}{\lambda K_\nu(\lambda a)} \right] A
\]

From the continuity of \( H_\phi \) at \( r = a \):

\[
y_B = \frac{j \omega \epsilon_0}{\epsilon_0} \left[ \frac{m_1^2 J_\nu'(\lambda a)}{J_\nu(\lambda a)} + \frac{m_2^2 K_\nu'(\delta a)}{\delta K_\nu(\delta a)} \right] \left[ \frac{1}{\lambda^2} + \frac{1}{\delta^2} \right] A
\]

Later we will see that we use one or the other of these relations depending on the nature of the mode description.
\[ E_z = A J_n(xr) e^{i \nu r} e^{-i \beta z} \]

\[ H_z = B J_n(xr) e^{i \nu r} e^{-i \beta z} \]

* The ratio \( \frac{B}{A} \) provides information on the projection of the mode. i.e. large \( A \) \( \rightarrow \) TM, since there is a large \( E_z \) component, large \( B \) \( \rightarrow \) TE, since there is a large \( H_z \) component.

* When \( \nu = 0 \) the modes are rotationally invariant since \( e^{i \nu r} = 1 \).

To obtain specific values for \( E_\phi, E_z, H_\phi, H_z \), substitute specific values for \( E_z, H_z \) when \( \nu > a \) and \( \nu < a \).

**Core region (\( r < a \)):**

\[ E_\phi = -\frac{j \beta^2}{\kappa^2} \left[ \frac{1}{\nu} A J_n(xr) + \frac{\omega \nu}{\beta r} B J_n(xr) \right] e^{i \nu r} e^{-i \beta z} \]

\[ E_z = -\frac{j \beta^2}{\kappa^2} \left[ \frac{j \pi}{\nu} A J_n(xr) - \frac{\omega \nu}{\beta r} B K_n(xr) \right] e^{i \nu r} e^{-i \beta z} \]

\[ H_\phi = -\frac{j \beta^2}{\kappa^2} \left[ B K_n(xr) - \frac{j \omega \nu}{\beta r} A J_n(xr) \right] e^{i \nu r} e^{-i \beta z} \]

\[ H_z = -\frac{j \beta^2}{\kappa^2} \left[ \frac{j \pi}{\nu} B J_n(xr) + \frac{\omega \nu}{\beta r} A K_n(xr) \right] e^{i \nu r} e^{-i \beta z} \]

**Cladding region (\( r > a \)):**

\[ E_\phi = \frac{i \beta^2}{\kappa^2} \left[ C_\delta K_n(\delta r) + \frac{i \omega \nu}{\beta r} D K_n(\delta r) \right] e^{i \nu r} e^{-i \beta z} \]

\[ E_z = \frac{i \beta^2}{\kappa^2} \left[ \frac{j \pi}{\nu} C K_n(\delta r) - \frac{\omega \nu}{\beta r} D_\delta K_n(\delta r) \right] e^{i \nu r} e^{-i \beta z} \]

\[ H_\phi = \frac{i \beta^2}{\kappa^2} \left[ D_\delta K_n(\delta r) - \frac{j \omega \nu}{\beta r} C K_n(\delta r) \right] e^{i \nu r} e^{-i \beta z} \]

\[ H_z = \frac{i \beta^2}{\kappa^2} \left[ \frac{j \pi}{\nu} D K_n(\delta r) + \frac{\omega \nu}{\beta r} C_\delta K_n(\delta r) \right] e^{i \nu r} e^{-i \beta z} \]

\( j K_n(\nu r) = \frac{d j_j(\nu r)}{d \nu} \)
The continuity expressions for $E_x$, $H_x$, $E_y$, $H_y$ can be written as a set of linear equations:

$$
\begin{pmatrix}
E_x(x) \\
H_x(x) \\
E_y(x) \\
H_y(x)
\end{pmatrix}
= \begin{pmatrix}
J_v(xa) & 0 & -K_v(\delta a) & 0 \\
0 & J_v(xa) & 0 & -K_v(\delta a) \\
-\frac{\beta_v}{a} J_v(xa) & -\frac{\beta_v}{a} J_v(xa) & \frac{\beta_v}{\delta^2} K_v(\delta a) & -\frac{\beta_v}{\delta} \delta K_v'(\delta a) \\
-\frac{j \omega \varepsilon}{K} J_v'(x) & \frac{\beta_v}{a} J_v(xa) & -\frac{j \omega \varepsilon}{\delta} \delta K_v'(\delta a) & \frac{\beta_v}{\delta^2} K_v(\delta a)
\end{pmatrix}
\begin{pmatrix}
A \\
B \\
C \\
D
\end{pmatrix}
$$

In order for the coefficients $A, B, C, D$ not to be all zeros, the determinant must be zero $\Delta = 0$.

Solving the expression for the determinant results in the characteristic equation for the cylindrical wave guide modes. [Specifically the modes for a step index cylindrical wave guide.]

General form of characteristic equation:

$$
\frac{\beta_v^2}{a^2} \left[ \frac{1}{\delta^2} + \frac{1}{\alpha^2} \right] = \left[ \frac{J_v'(xa)}{\delta J_v(xa)} + \frac{K_v'(\delta a)}{\delta K_v(\delta a)} \right].
$$

$$
\left[ \frac{\alpha^2 m^2}{\delta} \frac{J_v'(xa)}{\delta J_v(xa)} + \frac{\beta_v^2 m^2}{\delta} \frac{K_v'(\delta a)}{\delta K_v(\delta a)} \right]
$$
Resultant Characteristic Equation from matching at \( r = a \)

\[
\left( \frac{Kn'}{n} \right)^2 \left[ \frac{1}{a^2} + \frac{1}{x^2} \right]^2 = \left[ \frac{Jv'(xa)}{Kn(2xa)} + \frac{Kv'(ya)}{yKv(2ya)} \right].
\]

\[
\left[ \frac{\hbar^2 m^2_v Jv'(xa)}{Kn(2xa)} + \frac{\hbar^2 m^2_e K_v'(ya)}{yKv(2ya)} \right]
\]

A numerical or graphical solution is sought such that:

Left Hand Side = Right Hand Side

Note that \( \beta \) is the only unknown since

\[
y^2 = \beta^2 - (k_0 m_l)^2
\]

\[
x^2 = (k_0 m_l)^2 - \beta^2
\]

Specific Solutions:

If \( \nu = 0 \) the solutions for \( E_2 \) are rotationally invariant.

This can be seen by recalling \( e^{iv\varphi} = 1 \) : no change in \( \varphi \) with \( \nu = 0 \) the resulting modes are TE or TM and the characteristic equation becomes:

\[
[\frac{Jv'(xa)}{Kn(2xa)} + \frac{Kv'(ya)}{yKv(2ya)}] \left[ \frac{\hbar^2 m^2_v Jv'(xa)}{Kn(2xa)} + \frac{\hbar^2 m^2_e K_v'(ya)}{yKv(2ya)} \right] = 0
\]

To satisfy this relation either factor can be zero.

Recall the relations between the \( A \) and \( B \) coefficients.

1. If the first term is zero then \( A = 0 \) \( \text{TE mode} \)
   - If \( A = 0 \) then \( E_2 = 0 \) and the \( E \)-Field is transverse

2. If the second term is zero then \( B = 0 \) \( \text{TM mode} \)
   - If \( B = 0 \) then \( H_2 = 0 \) and the \( H \)-Field is transverse
Simplifying the Characteristic Equation

The roots of the first factor of equation A provide TE solutions.

The roots of the second factor of equation A indicate the TM solutions.

The derivatives in Eq. A can be reduced in form using the identities:

\[
\frac{J_x'}{KJ_x} = \frac{J_{x+1}}{KJ_x} + \frac{v}{x^2} \quad \Rightarrow \quad J_0' = -J_1 \quad \text{lower solution}
\]

\[
\frac{K_x'}{8K_x} = \frac{K_{x+1}}{8K_x} + \frac{v}{x^2} \quad \Rightarrow \quad K_0' = -K_1
\]

Consider the first factor (TE-modes):

\[-J_1(K_x) \quad \frac{K_1(8a)}{KJ_0(K_x)} = 0\]

The characteristic equation for the TE mode:

\[-J_1(K_x) = \frac{K_1(8a)}{KJ_0(K_x)}\]

The characteristic equation for the TM modes is determined from the second factor from Eq. A:

\[-\frac{k_0^2 m_1^2}{K} J_1(K_x) = \frac{k_0^2 m_1^2}{8} \frac{K_1(8a)}{K_0(8a)}\]

* The intersections give the values of Ka where a precise solution exists.

* Note however each time the denominator is zero, the LHS = 00 (on the LHS) and must pass through the exponentially decaying function on the RHS. i.e., the zeros of J_o indicate that a solution for a particular mode exists.
Example 5.1 Eigenvalues for the TE Modes in a Step-Index Fiber

Let's analyze a step-index circular fiber with a core index $n_{\text{core}} = 1.5$, a cladding index $n_{\text{clad}} = 1.45$, and a core radius $a = 5\mu m$. The wavelength of the light is $1.3\mu m$. We want to determine the allowed eigenvalues for $\beta$ for the TE modes. A simple Mathematica command evaluates and plots the two terms in Equation 5.33.

\[
k = 2 \, \pi \, \text{Pi}/(1.3 \, 10^3 \, (-4)); \quad a = 5 \, 10^3 \, (-4); \quad n_1 = 1.5; \quad n_2 = 1.45;
\]
\[
k_{\text{max}} = \text{Sqrt}[k^2 (n_1^2 - n_2^2)]; \quad \gamma = \text{Sqrt}[k_{\text{max}}^2 - k^2];
\]
\[
\text{Plot}[\{\text{BesselJ}[1, \kappa \, \alpha]/(\kappa \, \text{BesselJ}[0, \kappa \, \alpha]), \, \text{BesselK}[1, \gamma \, \alpha]/(\gamma \, \text{BesselK}[0, \gamma \, \alpha])\}, \{\kappa, 0, k_{\text{max}}\}];
\]

The graphical output is presented in Figure 5.6. As in previous chapters, we chose to plot the functions against the transverse wavevector $\kappa$, instead of against $\beta$. The plot extends from $\kappa = 0$ to $\kappa_{\text{max}}$, which is given by

\[
k_{\text{max}} \alpha = \sqrt{k_0^2 n_{\text{core}}^2 - k_0^2 n_{\text{clad}}^2} \quad (5.34)
\]

The $J_1(\kappa \alpha)/\kappa J_0(\kappa \alpha)$ term explodes to infinity at every root of $J_0(\kappa \alpha)$. Since the roots of $J_0(\kappa \alpha)$ occur (almost) periodically, the ratio $J_1/J_0$ regularly sweeps from $-\infty$ to $+\infty$. The $K_1/K_0$ term monotonically decreases as $\kappa$ increases.

![Figure 5.6 The eigenvalue equation is plotted against $\kappa$ for a waveguide with core index 1.5, cladding index 1.45, and wavelength 1.3\mu m.](image)

Every time the two lines cross in Figure 5.6, there is an allowed TE mode. In this case, three TE modes are allowed, with approximate $\kappa$ values of 7.000, 12.500, and 17.500 cm$^{-1}$. The exact values are easily found using a root-finding command. In Mathematica, the appropriate command is

\[
\text{FindRoot}[-\text{BesselK}[1, \gamma \alpha]/(\gamma \text{BesselK}[0, \gamma \alpha]) == \text{BesselJ}[1, \kappa \alpha]/(\kappa \text{BesselJ}[0, \kappa \alpha]), \{\kappa, 5200\}]
\]

The exact values for this example are $\kappa = 6.902, 12.549, \text{and } 17.795 \text{cm}^{-1}$. The corresponding values of $\beta$ can be determined from Equation 5.14.

\[
\beta^2 = (k_0 n_1)^2 - \kappa^2
\]
Hybrid Modes

These occur when $V \neq 0$.

In this case the modes are neither TE or TM but combinations of each type.

The allowed values of $\beta$ correspond to modes with finite values of $E_z$ and $H_z$.

These modes are referred to as the $EH$ and $HE$ modes.

Different Types of Modes:

- $A = 0 \rightarrow$ TE modes (no $E_z$ component)
- $B = 0 \rightarrow$ TM modes (no $H_z$ component)

Hybrid modes
- $A > B \rightarrow$ HE modes ($E_z > H_z$)
- $A < B \rightarrow$ EH modes ($E_z < H_z$)

The TE/TM modes may be thought of as meridional rays.

The hybrid HE/EH modes do not travel through the center of the core. i.e. more like slow rays.
• In Summary the common types of modes in circular fibers include:

1. *TE modes*: the $E_z$ field component = 0.
2. *TM modes*: the $H_z$ field component = 0.
3. *HE modes*: $E_z > H_z$.
4. *EH modes*: $H_z > E_z$.

• It is also possible to think of the TE and TM modes as **meridional rays** (*rays that pass through the axis of the fiber*) and EH and HE modes as **skew rays** (*rays that do not pass through the axis of the fiber*).
• For a given set of parameters $k_o$, $a$, $n_l$, and $n_z$ the characteristic equation can be solved numerically to determine the propagation constant $\beta$.

• In general there will be multiple solutions for each value of the azimuthal mode number $\nu$.

• The multiple solutions corresponding to a given value of $\nu$ are designated with a second mode number referred to as the radial mode number. This allows specifying a mode with $\beta_{vm}$.

• The characteristic equation contains a term with the ratio

\[
\frac{J_{\nu+1}(\kappa a)}{J_{\nu}(\kappa a)}
\]

This term $\to \infty$ at each root of $J_{\nu}(\kappa a)$

• In order for there to be at least one solution to the equation the argument $\kappa a$ must extend beyond the first of these roots.

• Each time $\kappa a$ increases beyond another root of $\frac{J_{\nu+1}(\kappa a)}{J_{\nu}(\kappa a)}$ another mode will be allowed.

• $\because$ the roots of the Bessel functions are good indicators of when a mode will be cut-off.

• It is possible to categorize the different modes as $TE_{vm}$, $TM_{vm}$, $EH_{vm}$, and $HE_{vm}$

• $TE$ and $TM$ can be though of as meridional rays propagating through the fiber. As before $TE$ refers to $E_z = 0$ and $TM$ when $H_z = 0$.

• Most of the time fields will have partial $TE$ and $TM$ characteristics.

• When $E_z > H_z$ the modes are referred to as $HE$ and when $E_z < H_z$ as $EH$. 
The $\text{TE}_{0m}$ modes are the $\kappa a > m^{th}$ root of $J_0(\kappa a)$.

The $\text{HE}_{1m}$ mode is the $\kappa a > m^{th}$ root of $J_1(\kappa a)$.

The $\text{EH}_{\nu m}$ mode is the $\kappa a > m^{th}$ root of $J_\nu(\kappa a)$, with the additional constraint that the first root is not 0.

The $\text{HE}_{\nu m}$ modes are given by:

$$\left(\frac{\varepsilon_{\text{core}}}{\varepsilon_{\text{clad}}} + 1\right) J_{\nu - 1}(\kappa a) = \frac{\kappa a}{\nu - 1} J_{\nu}(\kappa a)$$

This condition differs from the other three modes.

Figure 5.11(a) shows a plot of the first three Bessel functions, with notations on the cutoff points for a few modes. For example, if $\kappa a$ is greater than 2.405, then the $	ext{HE}_{21}$ mode is allowed.

Figure 5.11(b) is a plot of the relationship $2J_1(\kappa a) = \kappa a J_2(\kappa a)$, which is used to determine the cutoff for the $\text{HE}_{2m}$ mode.

The $\text{HE}_{21}$ mode is allowed when $\kappa a > 2.405$.

Cutoffs are predicted from the plots.

The $\text{TE}_{01}$, $\text{TM}_{01}$, and $\text{HE}_{21}$ modes will be allowed.
Example: Finding the number of TE and TM modes in a fiber.

Example 5.3 Number of TE Modes in a Step-Index Fiber

Consider a step-index fiber that has a core index $n_{core} = 1.45$, a cladding index $n_{clad} = 1.44$, and a core radius of $25 \mu m$. If the excitation wavelength is $1.5 \mu m$, how many TE and TM modes will exist in the waveguide?

Solution:

First calculate the normalized frequency for the fiber:

$$V = \frac{2\pi 25 \mu m}{1.5 \mu m} \sqrt{1.45^2 - 1.44^2}$$

(5.51)

$$= 17.802 \text{ cm}^{-1}$$

The zeros of the $J_0(ka) = 0$ occur at $2.504, 5.520, 8.654, 11.791, 14.931, 18.071$, etc. (See Appendix B, “Bessel Functions”.) Clearly, $V$ is larger than the first five roots, but is smaller than the sixth root at $18.071$. So five TE modes (and five TM modes) will be allowed in this waveguide at that wavelength.

There are two degrees of freedom (two dimensions)

- Two sets of index numbers describing the modes (we just saw that several solutions exist corresponding to each root.)
  $\beta_{nm} \rightarrow HE_{nm}$
  $\beta_{nm} \rightarrow TE_{nm}$

$V = \text{Angular mode number}$

$\beta_{nm} \rightarrow \text{Radial mode number}$

Moving out along the radius, the field changes twice.

Moving around the azimuth, the field changes twice.
Two separate field patterns with same $E+H$ orientation:

- **Intensity Distribution**
- **D in LP mode, only one $E+H$ field component**

**FIGURE 2-20**
The four possible transverse electric field and magnetic field directions and the corresponding intensity distributions for the $LP_{11}$ mode.

**FIGURE 2-21**
Composition of two $LP_{11}$ modes from exact modes and their transverse electric fields.
Figure 10.12. Mode set for a small V-value fiber, obtained using the coupler of Figure 10.11.

Figure 10.11. The prism-taper coupler for accessing all the guided modes of a fiber.
Figure 10.10. Intensity patterns of two LP modes, excited by means of a prism coupler. (a) An LP_{17,16} mode excited in a step-index fiber. (b) An LP_{25,25} mode excited in a graded-index fiber. Photos reproduced with permission of W. J. Stewart, Plesey Company Ltd., Carwell, England.
• The $HE_{11}$ mode corresponds to the lowest order mode. It is cut off only when the radius of the fiber goes to 0.

• In practice the modes represent spatial patterns of the fields that are stable within the fiber.

• The $HE_{11}$ is close to a Gaussian distribution when $V\# \sim 2.405$. An $HE_{12}$ appears as:

Another way that is often used to illustrate fiber modes and cut-off conditions is to plot the normalized propagation constant $b$ as a function of the $V\#$.

$$b = \frac{\beta / k_o - n_2}{n_1 - n_2} = \frac{\overline{n} - n_2}{n_1 - n_2}$$

$$\overline{n} = \beta / k_o$$

$\overline{n}$ is the mode index or effective index of the fiber and each fiber mode propagates with an effective index $n_1 > \overline{n} > n_2$. A mode is no longer guided when $\overline{n} \leq n_2$.

• As discussed earlier a fiber with a large $V\#$ (i.e. $V > 10$) supports a large number of modes. In this case the number of modes $M$ can be approximated by

$$M \approx V^2 / 2$$
**Linearly Polarized Modes**

When the refractive index of the core $n_1 = n_2$ several sets of modes are degenerate and allow these modes to be grouped together and be considered as one mode.

To see how this arises consider the general characteristic equation:

$$
\left( \frac{\beta \nu}{a} \right)^2 \left[ \frac{1}{\gamma^2} + \frac{1}{\kappa^2} \right]^2 = \left[ \frac{J'_v(\kappa a)}{\kappa J_v(\kappa a)} + \frac{K'_v(\gamma a)}{\gamma K_v(\gamma a)} \right] \left[ \frac{k_0^2 n_1^2 J'_v(\kappa a)}{\kappa J_v(\kappa a)} + \frac{k_0^2 n_2^2 K'_v(\gamma a)}{\gamma K_v(\gamma a)} \right]
$$

When $n_1 = n_2$ the *weakly guiding approximation* is valid. Typically $\Delta n = n_1 - n_2 \sim (0.001-0.005)$. If it is assumed that $\Delta n = 0$ the error in predicting the value of the propagation vector will only be on the order of 0.10%.

Assuming that $n_1 = n_2$ the characteristic equation (CE) reduces to:

$$
\left( \frac{\beta \nu}{a} \right)^2 \left[ \frac{1}{\gamma^2} + \frac{1}{\kappa^2} \right]^2 = \left[ \frac{J'_v(\kappa a)}{\kappa J_v(\kappa a)} + \frac{K'_v(\gamma a)}{\gamma K_v(\gamma a)} \right]^2 k_0^2 n_1^2.
$$

Also note that $\beta^2 = (k_0 n_1)^2$ therefore this term can be cancelled yielding

$$
\nu \left[ \frac{1}{(a \gamma)^2} + \frac{1}{(a \kappa)^2} \right] = \left[ \frac{J'_v(\kappa a)}{\kappa J_v(\kappa a)} + \frac{K'_v(\gamma a)}{\gamma K_v(\gamma a)} \right] \text{ for the CE.}
$$

Using the recurrence relation for Bessel functions the CE can be re written as two separate equations (due to the $\pm$ forms of the recurrence relations) as

$$
\frac{J_{v+1}(\kappa a)}{\kappa J_v(\kappa a)} = -\frac{K_{v+1}(\gamma a)}{\gamma K_v(\gamma a)} \quad (1) \text{ obtained from the } + \text{ sign in the recurrence relation}
$$

result in EH modes.

$$
\frac{J_{v-1}(\kappa a)}{\kappa J_v(\kappa a)} = \frac{K_{v-1}(\gamma a)}{\gamma K_v(\gamma a)} \quad (2) \text{ obtained from the } - \text{ sign in the recurrence relation}
$$

correspond to the HE modes. $\nu = 1 \rightarrow 0$

The CE for the HE modes with $\nu \geq 2$ can be written as
\[
\frac{J_{\nu-1}(\kappa a)}{\kappa J_{\nu-2}(\kappa a)} = \frac{-K_{\nu-1}(\gamma a)}{\gamma K_{\nu-2}(\gamma a)}.
\]

The integer \(\nu\) gives the mode number in the azimuthal direction \(\varphi\) and \(m \geq 1\) is a mode number in the radial direction. \(m\) represents the \(m^{th}\) solution of each CE. A new parameter \((l)\) is defined as

\[
l = \nu + 1
\]

where \(l = 1\) corresponds to the TE and TM modes,

\[
l = \nu + 1\] to the EH modes; and \(l = \nu - 1\) to the HE modes.

Using the parameter \(l\) the CE can be written in the unified form as

\[
\frac{J_{l}(\kappa a)}{\kappa J_{l-1}(\kappa a)} = \frac{-K_{l}(\gamma a)}{\gamma K_{l-1}(\gamma a)}.
\]

Since several HE modes have the same LP eigenvalues. Therefore the LP modes are approximate modes classified by the eigenvalues.

Three types of modes correspond to \(l = 1\): TE\(_0\), TM\(_0\), and HE\(_2\) satisfy the same characteristic equation.

Combinations EH\(_{l-1,m}\) and HE\(_{l+1,m}\) corresponding to \(l \geq 2\) also satisfy the same CE.

### Table 3.2: Comparison of LP modes with conventional modes.

<table>
<thead>
<tr>
<th>LP mode ((\ell \geq 1))</th>
<th>Conventional mode ((\ell \geq 1))</th>
<th>Dispersion equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP(_{0\ell}) mode ((\ell = 0))</td>
<td>HE(_1) (\alpha) mode</td>
<td>(\frac{J_0(u)}{u J_1(u)} = \frac{K_0(w)}{w K_1(w)})</td>
</tr>
<tr>
<td>LP(_{1\ell}) mode ((\ell = 1))</td>
<td>TE(_0) (\alpha) mode</td>
<td>(\frac{J_1(u)}{u J_0(u)} = \frac{K_1(w)}{w K_0(w)})</td>
</tr>
<tr>
<td>LP(_{m\ell}) mode ((\ell \geq 2))</td>
<td>EH(_{l-1\ell}) (\alpha) mode</td>
<td>(\frac{J_{m-1}(u)}{u J_{m-1}(u)} = \frac{K_{m-1}(w)}{w K_{m-1}(w)})</td>
</tr>
<tr>
<td>LP(_{m\ell}) mode ((\ell \geq 2))</td>
<td>HE(_{m+1\ell}) (\alpha) mode</td>
<td>(\frac{J_{m+1}(u)}{u J_{m+1}(u)} = \frac{K_{m+1}(w)}{w K_{m+1}(w)})</td>
</tr>
</tbody>
</table>
Solutions to LP Mode Conditions:
Solving the characteristic equation for the step index fiber given in Table 3.2 in conjunction with the requirement that

\[ V^2 = (\kappa a)^2 + (\gamma a)^2 \]

provides the values for the allowed modes. Note that in this form the relation between \( \kappa a \) and \( \gamma a \) are plotted (See Fig 3.2).

Note that the LP_{01} is the lowest order mode and corresponds to the HE_{11} mode.

(Note for comparison with other texts \( u = \kappa a \) and \( w = \gamma a \).)
<table>
<thead>
<tr>
<th>Designation</th>
<th>Designation</th>
<th>Electric Field Pattern</th>
<th>Intensity Distribution of $E_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP$_{01}$</td>
<td>HE$_{11}$</td>
<td><img src="image1" alt="Electric Field Pattern" /></td>
<td><img src="image2" alt="Intensity Distribution" /></td>
</tr>
<tr>
<td>TE$_{01}$</td>
<td></td>
<td><img src="image3" alt="Electric Field Pattern" /></td>
<td><img src="image4" alt="Intensity Distribution" /></td>
</tr>
<tr>
<td>LP$_{11}$</td>
<td>TM$_{01}$</td>
<td><img src="image5" alt="Electric Field Pattern" /></td>
<td><img src="image6" alt="Intensity Distribution" /></td>
</tr>
<tr>
<td>HE$_{21}$</td>
<td></td>
<td><img src="image7" alt="Electric Field Pattern" /></td>
<td><img src="image8" alt="Intensity Distribution" /></td>
</tr>
<tr>
<td>LP$_{21}$</td>
<td>EH$_{11}$</td>
<td><img src="image9" alt="Electric Field Pattern" /></td>
<td><img src="image10" alt="Intensity Distribution" /></td>
</tr>
<tr>
<td>HE$_{31}$</td>
<td></td>
<td><img src="image11" alt="Electric Field Pattern" /></td>
<td><img src="image12" alt="Intensity Distribution" /></td>
</tr>
</tbody>
</table>

Figure 3.2: Electric field vectors and intensity profiles of LP modes and conventional modes.

![Graph](image13)

**Figure 3.3:** $u-w$ relation in the step-index fiber.
Total Power in a mode is determined by integration of the
Propagating Vector

\[ \langle s_2 \rangle = \frac{1}{2} \text{Re}(E \times H^*) \cdot \hat{n} \]

\[ P_{\text{core}} = \frac{1}{2} \int_0^\infty \int_0^{2\pi} \nu \left( E_x H_y^* - E_y H_x^* \right) d\phi d\nu \]

\[ P_{\text{clad}} = \frac{1}{2} \int_0^\infty \int_0^{2\pi} \nu \left( E_x H_y^* - E_y H_x^* \right) d\phi d\nu \]

\[ \frac{P_{\text{core}}}{P_{\text{total}}} = 1 - \frac{(Ka)^2}{V^2} \left[ 1 - \frac{K_a^2(ka)}{K(ka) K(ka)} \right] \]

\[ P_{\text{clad}} = 1 - \frac{P_{\text{core}}}{P_{\text{total}}} \]

\[ P_{\text{total}} \sim \text{power in mode } \nu \]

Example:

SI fiber with \( a = 25 \mu m \), \( m = 1.48 \), \( \beta = 0.01 \), \( \lambda = 0.85 \mu m \)

\[ V = 39 \]

\( \nu \) \( \sim \) 6/16 modes \( (N = (\frac{V}{16})^4) \)

Assume every mode is excited with same amount of power.

Approximate Formula

\[ \frac{P_{\text{clad}}}{P_{\text{total}}} = \frac{4}{3} \frac{1}{N} \]

\( \sim 5.4\% \) of power is in cladding.

If \( \beta \) is reduced to 0.003 to reduce signal degradation

Then \( V = 21.4 \) and \( \frac{P_{\text{clad}}}{P_{\text{total}}} \sim \frac{4}{3} \frac{1}{N} \sim 9.8\% \) i.e. about 9.8\%

For single-mode fiber:

(See graph in figure)

From figure at \( V = 1 \) : \( \frac{P_{\text{clad}}}{P_{\text{total}}} \approx 70\% \)

\( V = 2.405 \) : \( \frac{P_{\text{clad}}}{P_{\text{total}}} \approx 15\% \)

This power distribution will affect the coupling efficiency of
different types of couplers and splitters.
Note: For large number of modes, the power can be computed from

\[ \frac{P_{12d}}{P_{tot}} = \frac{4}{3} \frac{1}{N} \]

This assumes all modes are excited with the same power and is illuminated with an incoherent source.

For cases with intermediate \( N \) to use the expression in Gloge's article.
Figure 2-18
Plots of the propagation constant (in terms of $\beta/k$) as a function of $V$ for a few of the lowest-order modes.

Figure 2-19
Plots of the propagation constant $b$ as a function of $V$ for various $LP_{m}$ modes. (Reproduced with permission from Glege.)

\[ b = \frac{a^2 \beta^2}{V^2} = \frac{(\beta/k)^2 - \mu_0}{\mu_1^2 - \mu_2^2} \]

Table 2-2
Composition of the lower-order linearly polarized modes

<table>
<thead>
<tr>
<th>LP-mode designation</th>
<th>Traditional-mode designation and number of modes</th>
<th>Number of degenerate modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$LP_{01}$</td>
<td>$HE_{11} \times 2$</td>
<td>2</td>
</tr>
<tr>
<td>$LP_{11}$</td>
<td>$TE_{01}, TM_{01}, HE_{21} \times 2$</td>
<td>4</td>
</tr>
<tr>
<td>$LP_{21}$</td>
<td>$EH_{11} \times 2, HE_{31} \times 2$</td>
<td>4</td>
</tr>
<tr>
<td>$LP_{31}$</td>
<td>$HE_{12} \times 2$</td>
<td>2</td>
</tr>
<tr>
<td>$LP_{12}$</td>
<td>$EH_{21} \times 2, HE_{41} \times 2$</td>
<td>4</td>
</tr>
<tr>
<td>$LP_{22}$</td>
<td>$TE_{02}, TM_{02}, HE_{22} \times 2$</td>
<td>4</td>
</tr>
<tr>
<td>$LP_{32}$</td>
<td>$EH_{31} \times 2, HE_{51} \times 2$</td>
<td>4</td>
</tr>
<tr>
<td>$LP_{13}$</td>
<td>$EH_{12} \times 2, HE_{32} \times 2$</td>
<td>4</td>
</tr>
<tr>
<td>$LP_{23}$</td>
<td>$HE_{13} \times 2$</td>
<td>2</td>
</tr>
<tr>
<td>$LP_{43}$</td>
<td>$EH_{41} \times 2, HE_{61} \times 2$</td>
<td>4</td>
</tr>
</tbody>
</table>
Fig. 8.5: Variation of the normalized propagation constant $b$ with normalized frequency $V$ for a step index fiber corresponding to some low-order modes. The cutoff frequencies of $LP_{2m}$ and $LP_{0,m+1}$ modes are the same. [Adapted from Gloge (1971)].

for $2.4048 < V < 3.8317$, only $LP_{01}$ and $LP_{11}$ modes;
for $3.8317 < V < 5.1356$, only $LP_{01}$, $LP_{11}$, $LP_{21}$, and $LP_{02}$ modes;
for $5.1356 < V < 5.5201$, only $LP_{01}$, $LP_{11}$, $LP_{21}$, $LP_{02}$, and $LP_{31}$

Fig. 8.6: Radial intensity distributions (normalized to the same power) of some low-order modes in a step index fiber for $V = 8$. Note that the higher order modes have a greater fraction of power in the cladding.
FIGURE 2-22
Fractional power flow in the cladding of a step-index optical fiber as a function of \( V \). When \( \nu \neq 1 \), the curve numbers \( \nu m \) designate the \( HE_{\nu+1,m} \) and \( EH_{\nu-1,m} \) modes. For \( \nu = 1 \), the curve numbers \( \nu m \) give the \( HE_{2m}, \) \( TE_{0m}, \) and \( TM_{0m} \) modes. (Reproduced with permission from Gloge.²⁰)
**Linearly Polarized Modes:**

- Since the refractive index difference between the core and cladding is so small *degeneracy* exists between modes. \( \Delta n \ll 1 \)

- One way to think of this is that *groups of modes travel together* with the *same velocity*.

- These groups of modes are collectively called the *linearly polarized* (LP) modes.

- The LP\(_{01}\) mode is the HE\(_{11}\) mode. Other groups of LP modes are:

  \[ \text{LP}_{1m} \rightarrow \text{sum of } \text{TE}_{om}, \text{TM}_{om}, \text{and HE}_{2m} \]

  \[ \text{LP}_{vm} \rightarrow \text{sum of HE}_{v+1,m}, \text{EH}_{v-1,m} \text{ modes} \]

  \[ \text{LP}_{0m} \rightarrow \text{HE}_{1m} \text{ mode only (special case)} \]

---

**Mode Profile of the Lowest Order Mode (HE\(_{11}\) or LP\(_{01}\)):**

The HE\(_{11}\) or LP\(_{01}\) mode has no cut-off.

The transverse field of this mode is described by the \( J_0 \) Bessel function in the core region of the fiber.

In many cases this can be approximated by a Gaussian beam where the field \( E(r) \) is given by

\[
E(r) = E_0 \exp \left[-\left(\frac{r}{w}\right)^2\right]
\]

\( w \) is the beam waist of the Gaussian. For the best fit between a Gaussian function and the Bessel function in the core

\[
w = a \left[ 0.65 + 1.619V^{-1.5} + 2.87V^{-6} \right]
\]
Satisfying this condition gives about 96% overlap between the Gaussian and the Bessel function mode profiles.

At the cut-off condition $V = 2.405$

$$w \approx 1.1a$$

The distance between the $1/e$ pts of the transverse amplitude is referred to as the *mode field diameter*.

In the range

$$0.8 \lambda_c \rightarrow 2 \lambda_c$$

$$\lambda_c = \frac{2\pi a}{2.405} \sqrt{n_1^2 - n_2^2}$$

the degree of overlap can vary significantly as shown in the figure.
Fig. 1.9 Shape of the guided intensity $I(r)$ for various values of the normalized frequency $V$. In all cases the total power guided by each mode is constant. The solid and dashed lines are obtained from the Bessel functions, whereas the dotted lines correspond to the appropriate Gaussian approximation, with the same total power. Note the changes in horizontal and vertical scales.