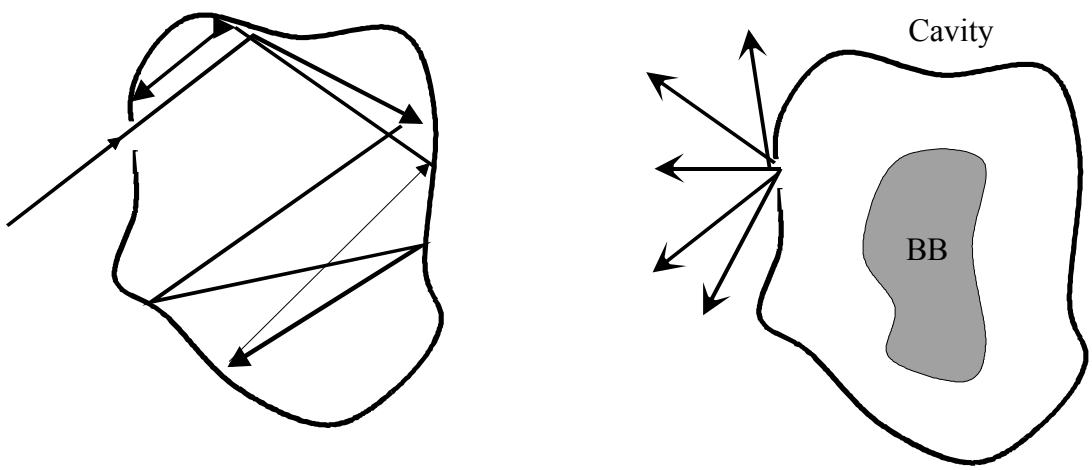
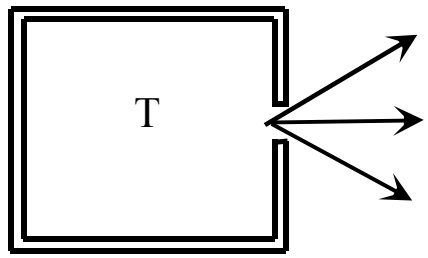
Black Body Radiation and Radiometric Parameters:

- All materials absorb and emit radiation to some extent. A *blackbody* is an idealization of how materials emit and absorb radiation. It can be used as a reference for real source properties.
- An ideal blackbody absorbs all incident radiation and does not reflect. This is true at all wavelengths and angles of incidence.
- Thermodynamic principals dictates that the BB must also radiate at all λ's and angles.
- The basic properties of a BB can be summarized as:
  1. Perfect absorber/emitter at all λ's and angles of emission/incidence.

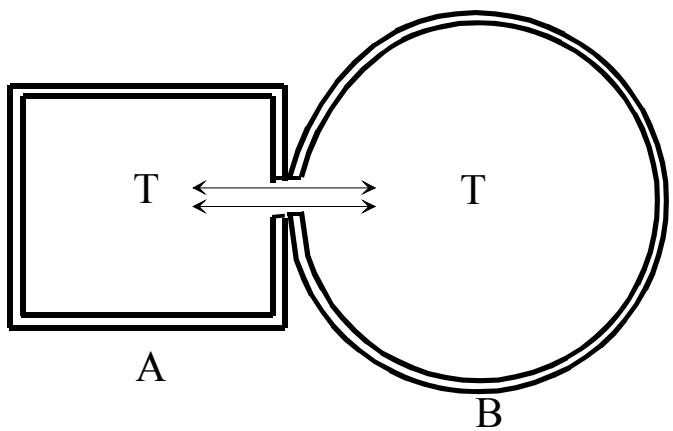


  2. The total radiant energy emitted is only a function of the BB temperature.
  3. Emits the maximum possible radiant energy from a body at a given temperature.

4. The BB radiation field does not depend on the shape of the cavity. The radiation field must be homogeneous and isotropic.



If the radiation going from a BB of one shape to another (both at the same T) were different it would cause a cooling or heating of one or the other cavity. This would violate the 1st Law of Thermodynamics.

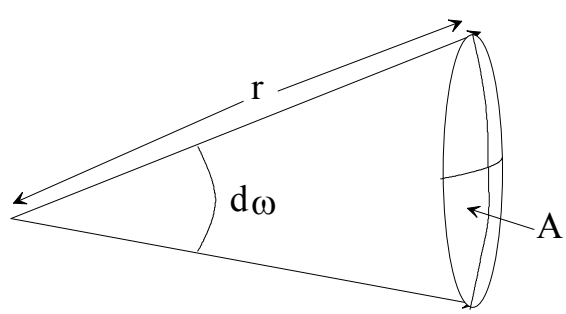


<u>Radiometric Parameters:</u>

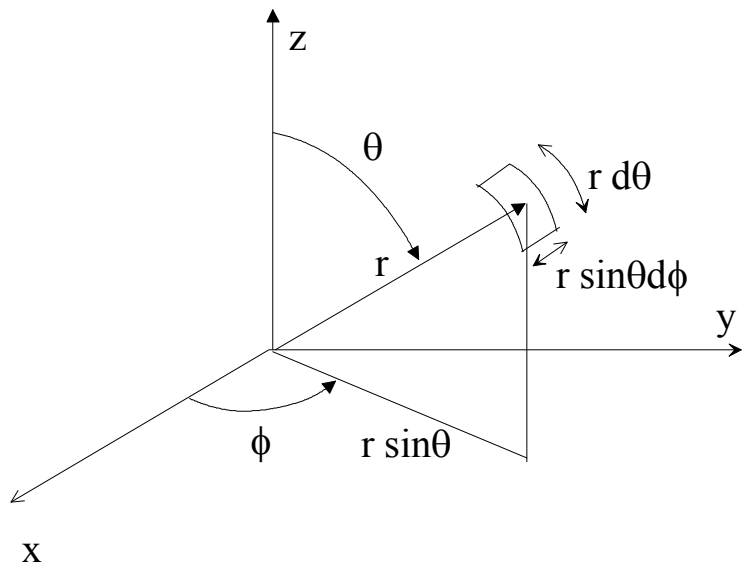
1. Solid Angle

$$d\omega = \frac{dA}{r^2}$$

where dA is the surface area of a segment of a sphere surrounding a point.



r is the distance from the point on the source to the sphere. The solid angle looks like a cone with a spherical cap.



An element of area of a sphere

$$dA = r^2 \sin\theta d\theta d\varphi$$

Therefore

$$d\omega = \sin\theta d\theta d\varphi$$

The full solid angle surrounding a point source is:

$$\int d\omega = \int_{\theta=0}^{\pi} \sin\theta d\theta \int_{\varphi=0}^{2\pi} d\varphi$$

$$= 2\pi \left\{ -\cos\theta \Big|_0^{\pi} \right\}$$

$$= 4\pi$$

Or integrating to other angles θ < π:

$$\Omega_\theta = 2\pi \left[1 - \cos\theta \right]$$

The unit of solid angle is steradian.

2. *Radiant Flux Φ and Energy Density:*

Radiant energy Q (J);

Energy flux is the rate of radiant energy transferred from one point or surface. Energy flux (Φ) is measured in watts. Can be spectral Φ_λ or a total over all wavelengths. (Note that *Flux* is the same as optical power.) The *energy density (u)* is the energy per unit volume.

$$\Phi = dQ / dt$$

$$u = dQ / dV$$

Note that Φ is the total flux integrated over all wavelengths. It consists of the integrated spectral flux or power with:

$$\Phi = \int_0^\infty \Phi(\lambda) d\lambda$$

$$\Phi = \int_0^\infty \Phi(\nu) d\nu$$

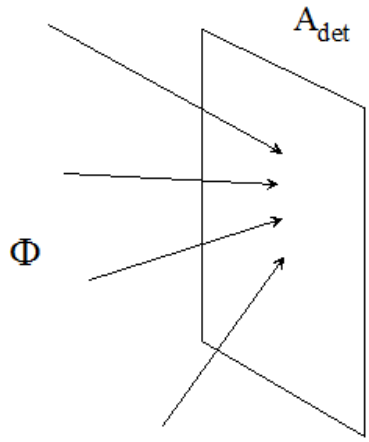
Note that $\Phi(\lambda) \neq \Phi(\nu)$ but $\Phi(\lambda) d\lambda = \Phi(\nu) d\nu$ since this represents an equal increment of power. The scaling factor between units is found by using the relation:

$$\lambda = c / \nu$$

$$\therefore d\nu = -\frac{c}{\lambda^2} d\lambda$$

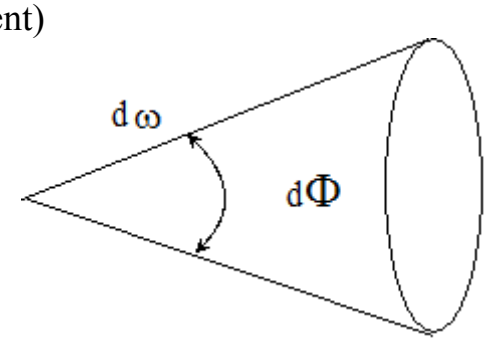
$$\Phi(\lambda) = \Phi(\nu) \frac{c}{\lambda^2}$$

3. *Irradiance:* The power per unit area illuminating a collection or detection surface.



$$E = \frac{d\Phi}{dA_{\text{det}}}$$

4. Spectral Intensity: The power emitted per unit solid angle from a source. (Wavelength dependent)



$$I(\lambda) = \frac{d\Phi(\lambda)}{d\omega}$$

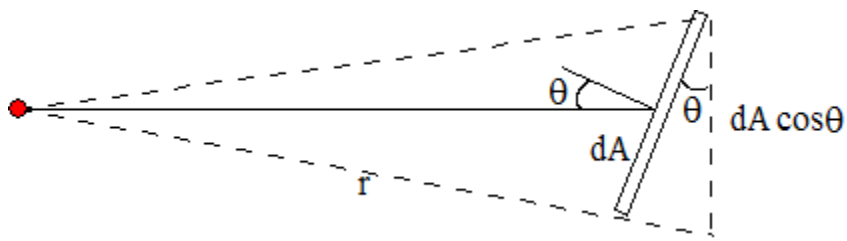
where dω is an increment of solid angle. Note that in physics *intensity* refers to the magnitude of the Poynting vector of an EM field. This interpretation resembles *irradiance* as defined here.

Note that if we consider a receiving surface element dA, the solid angle subtended by this surface relative to a point source is $d\Omega = dA\cos\theta / r^2$. In this case the irradiance is related to the intensity by:

$$E = \frac{d\Phi}{dA}$$

$$I = \frac{d\Phi}{\cos\theta dA / r^2}$$

$$\therefore E = \frac{I\cos\theta}{r^2}$$

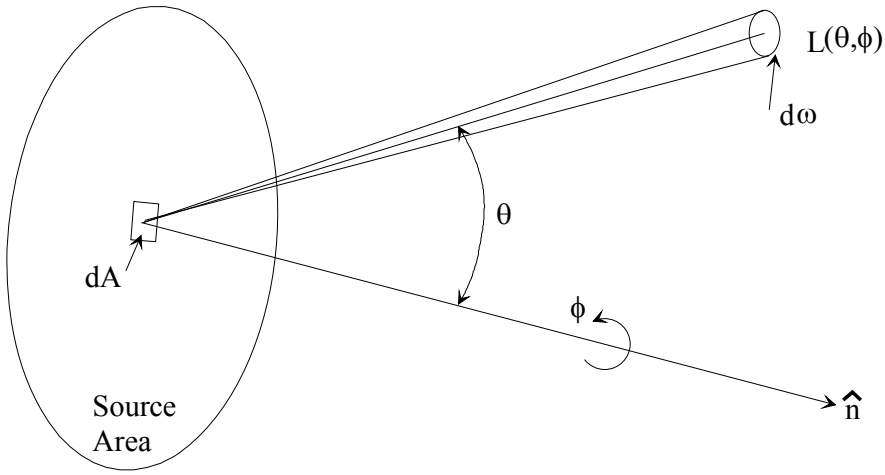


5. Spectral Emitance: The *power* per *unit source area* emitted from a source. (Wavelength dependent.)

$$M(\lambda) = \frac{d\Phi(\lambda)}{dA_{src}}$$

6. Spectral Radiance: The *power emitted* per *unit of projected source area per solid angle*. (Wavelength dependent.)

$$L(\lambda) = \frac{d\Phi(\lambda)}{\cos\theta \quad dA_{src} d\omega} = \frac{I(\lambda)}{\cos\theta \quad dA_{src}}$$



Spectral intensity, emittance, and radiance are terms referring to an optical source. They can also be integrated over all wavelengths to obtain the total intensity, emittance, and radiance from the source. Radiance is the most general of the source radiometric terms.

The radiant energy dQ(λ) emitted from an area dA over time dt, wavelength interval dλ, and solid angle dω in the direction θ, ϕ is related to the monochromatic radiance by:

$$dQ(\lambda) = L(\lambda)\cos\theta\, dA\, d\omega\, d\lambda\, dt \,.$$

Note that this represents the energy emitted from the source element dA at an angle θ relative to the surface normal $\hat{n}$. $\cos\theta\, dA$ is the projected area of the source.

The optical flux or power emitted from an element of source area dA can be estimated using the approximate relation:

$$\Delta\Phi(\lambda) = L(\lambda)\cos\theta\, \Delta\omega\, \Delta A.$$

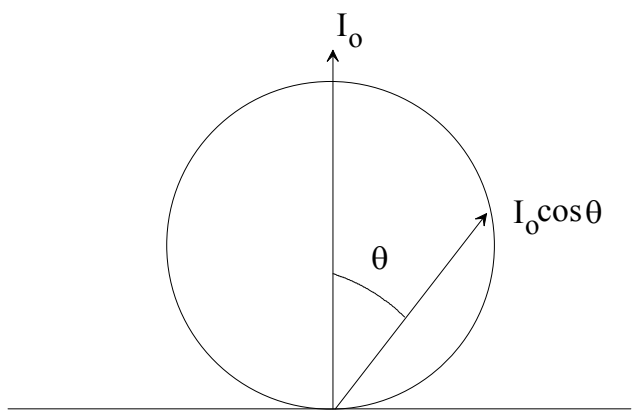
where $\Delta\Phi(\lambda) \approx \dfrac{\Delta Q(\lambda)}{dt}$

A *Lambertian* source emits light with the following characteristic:

$$I(\theta) = \int_{src} L_o \cos\theta dA$$

$$I(\theta) = I_o \cos\theta \quad ,$$

with $I_o = \int_{src} L_o dA$.



A BB acts as a *Lambertian* emitter. For a BB the emittance and radiance are related according to:

$$\begin{aligned} M_\lambda &= \int L_\lambda \cos\theta d\Omega \\ &= 2\pi L_\lambda \int_0^{\pi/2} \sin\theta \cos\theta d\theta \\ &= 2\pi L_\lambda \int_0^1 \cos\theta d(\cos\theta) \\ &= \pi \ L_\lambda \end{aligned}$$

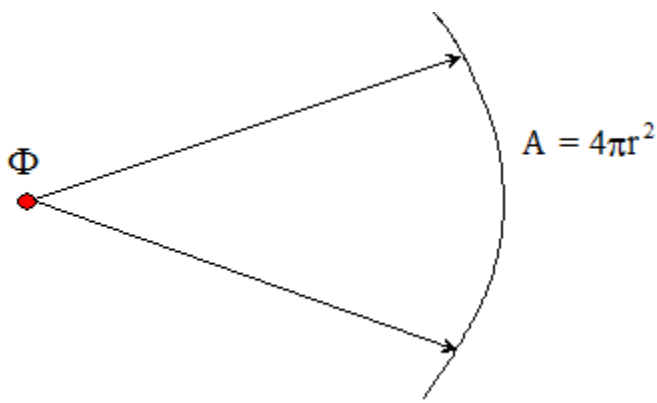
Note that the definition of the solid angle: $d\Omega = \sin\theta d\theta d\varphi$ is used with $0 < \varphi < 2\pi$.

<u>Relations between Parameters:</u>

From a point source: $$I(W/sr) = \frac{\Phi}{4\pi}$$

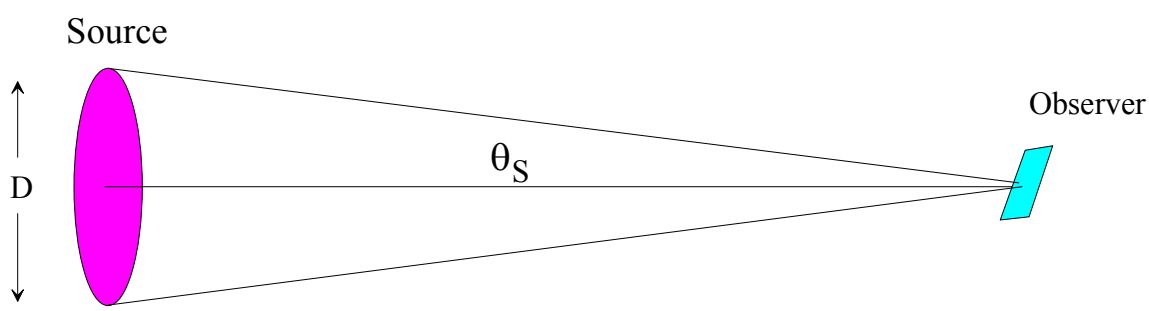
At a distance r from the point source the irradiance on a plane perpendicular to r is:

$$E = \frac{I}{r^2} = \frac{\Phi}{4\pi r^2}.$$



The irradiance was obtained by integrating over the area surrounding a point that emits power Φ. It can be seen that the power decreases as $1/r^2$ which is a well known radiation property.

Even if a source has a finite diameter it is still possible to use the inverse-square distance relation to approximate the power emitted from a source



provided that the ratio of the source diameter-to distance is small enough.

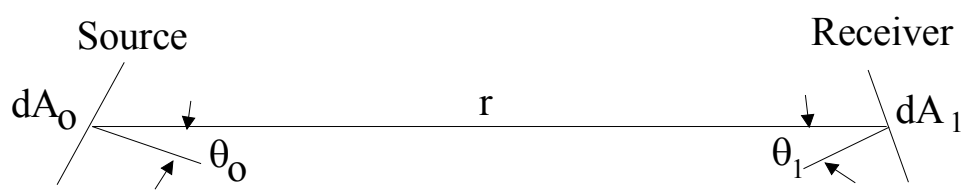
Src distance/diameter ratio	Subtended Angle	Inverse Sq. Law Error
1.6	~32°	~10%
5	~11.3^o	~1%
10	~5.7^o	~0.25%
16	~3.6^o	~0.1%

Note that the limit of angular resolution for the human eye is ~ 0.1^o. Therefore for visual applications a ratio of distance/source diameter of 16 is adequate for approximating an extended source as a point source. The angle subtended by the sun is 0.52^o therefore it is definitely a point source when viewed from the Earth.

Conservation of Radiance:

The *radiance theorem* is an important law of radiometry and states that *radiance is conserved* with propagation through a lossless optical system.

The radiance measured at the source and receiver is compared.



Consider a *source area* dA_o and *receiver area* dA_1 separated by a distance r.

The corresponding solid angles are:

$d\Omega_o$ = the solid angle subtended by dA_1 at dA_o

$$d\Omega_o = \frac{dA_1 \cos\theta_1}{r^2}$$

$d\Omega_1$ = the solid angle subtended by dA_o at dA_1

$$d\Omega_1 = \frac{dA_0 \cos\theta_0}{r^2}$$

If L_0 is the radiance of the radiation field measured at dA_0 in the direction of dA_1, the flux transferred from dA_0 to dA_1 is:

$$d^2\Phi = L_0 \left(dA_0 \cos\theta_0\right)\left(d\Omega_0\right)$$

Similarly the flux transferred from dA_1 to dA_0 is:

$$d^2\Phi = L_1 \left(dA_1 \cos\theta_1\right)\left(d\Omega_1\right).$$

The radiance L_1 measured at dA_1 is therefore:

$$L_1 = \frac{d^2\Phi}{dA_1 \cos\theta_1 d\Omega_1}$$

And since the flux originates from dA_o:

$$L_1 = \frac{d^2\Phi}{dA_1 \cos\theta_1 d\Omega_1} = \frac{L_0 dA_0 \cos\theta_0 d\Omega_0}{dA_1 \cos\theta_1 d\Omega_1}$$

$$\therefore L_1 = L_0$$

This implies that the radiance of the <u>source</u> is the *same* regardless of where it is measured and is conserved.

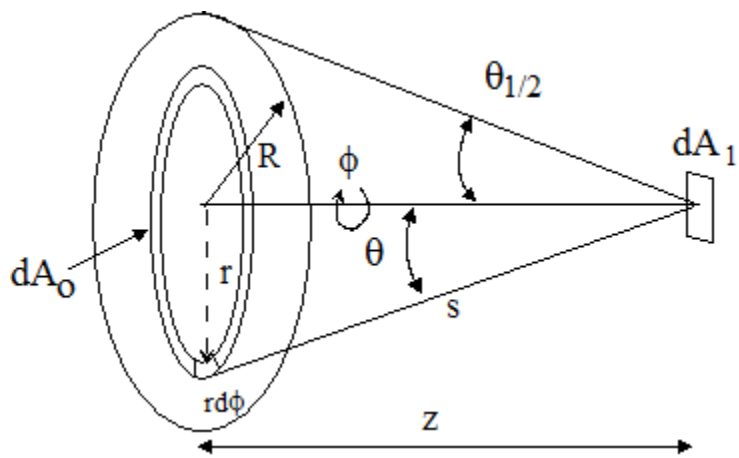
Another interesting result can be obtained by re-writing the expression:

$$d^2\Phi = L_0\left(dA_0\cos\theta_0\right)\left(d\Omega_0\right) = L_0\left(dA_0\cos\theta_0\right)\left(\frac{dA_1\cos\theta_0}{r^2}\right)$$
$$= L_0\left(dA_1\cos\theta_1\right)\left(d\Omega_1\right)$$

The result shows that the *transmitted flux* can be obtained by taking the projected area and solid angle *product* at the *source* or *receiver*.

This allows using a perspective either from the *source* or *receiver* to perform an analysis.

Lambertian Disk Source:
It is desired to compute the *irradiance* from a disk Lambertian source of radius R and uniform radiance L at an area element dA_1 that is parallel to the surface of the disk and is located axially at a distance z from the center of the disk.



An *annular area element* on the surface on the source is given by:

$$dA_0 = 2\pi z^2 \frac{\sin\theta d\theta}{\cos^3\theta}$$

The element of *solid angle* subtended by dA_1 from any point on dA_0 is given by:

$$d\Omega_0 = \frac{dA_1\cos\theta}{\left(z/\cos\theta\right)^2}$$

The *flux transferred* from dA_0 to dA_1 is given according to the definition of radiance as:

$$d^2\Phi = 2\pi L dA_1 \sin\theta \cos\theta d\theta$$

The irradiance at dA_1 becomes:

$$E = \frac{d\Phi}{dA_1} = \int_0^{\theta_{1/2}} 2\pi L \sin\theta \cos\theta d\theta$$

$$= \pi L \sin^2 \theta_{1/2} = \pi L \left(\frac{R^2}{R^2 + z^2} \right)$$

with

$$\theta_{1/2} = \tan^{-1}\left(\frac{R}{z}\right)$$

Note that when z << R the irradiance approaches a value of πL. This is the same value found previously for the *radiant exitance* of a Lambertian source.

When z >> R the irradiance approaches

$$E \rightarrow \pi R^2 L / z^2$$

in this case the irradiance observes an *inverse square law*.

At very large z the source looks like a *point source* with the intensity of

$$I = \pi R^2 L$$

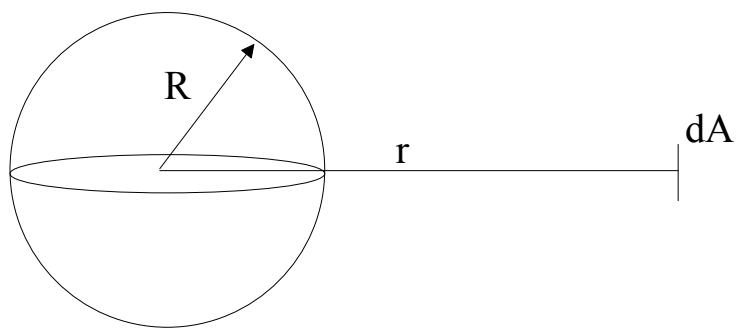
and the irradiance for a point source is:

$$E = \pi L\left(\frac{R^2}{R^2 + z^2}\right) \rightarrow \frac{\pi R^2 L}{z^2}$$

$$= \frac{I}{z^2}$$

which illustrates the inverse square law dependence.

Example:
Spherical Lambertian Source:
Consider the irradiance on an area dA located a distance r from the center of a spherical Lambertian source with uniform radiance L. To determine this we will use the symmetry of the situation rather than integrating over the surface of the source.



The radiant *exitance* of a *Lambertian* source is given by:

$$M = \pi L$$

The total flux emitted by the source is

$$\Phi = 4\pi^2 R^2 L$$

At a distance r from the center of the source it radiates uniformly over an area $4\pi r^2$.

Therefore the irradiance at a distance r is given by:

$$E = \frac{\pi R^2 L}{r^2}.$$

The irradiance is seen to follow an *inverse square law* at a distance r. The corresponding intensity becomes:

$$I = \frac{\Phi}{4\pi} = \pi R^2 L.$$

An observer at dA looking *back* at the *source* will see a *uniform disk* with half angle:

$$\theta_{1/2} = \sin^{-1}\left(\frac{R}{r}\right).$$

<u>Planck's Radiation Law:</u>

The spectral radiant exitance from a BB is given by:

$$M_\nu = \frac{2\pi h\nu^3}{c^2}\left[\frac{1}{\exp\left(h\nu / kT\right) - 1}\right]\left(\frac{W}{m^2 - Hz}\right)$$

$$M_\lambda = \frac{2\pi hc^2}{\lambda^5}\left[\frac{1}{\exp\left(hc / \lambda kT\right) - 1}\right]\left(\frac{W}{m^2 - m}\right)$$

Note that due to the relation between λ and ν

$$M_\nu \neq M_\lambda$$

but

$$M_\nu d\nu = M_\lambda d\lambda$$

$$\nu = \frac{c}{\lambda}$$

$$d\nu = -\frac{c}{\lambda^2} d\lambda$$

<u>Stefan-Boltzmann Law:</u>

The total emittance from a BB is only a function of the BB temperature. It can be found by integrating M_λ over all wavelengths.

$$M = \sigma T^4 (W / m^2)$$

The Stefan-Boltzmann constant:

$$\sigma_s = \frac{2\pi^5 k^4}{15 c^2 h^3} = 5.673 \times 10^{-8} \left(\frac{W}{m^2 - K^4} \right)$$

Boltzmann Constant: $k = 1.38 \times 10^{-23} J / K$

<u>Emissivity: ε_λ</u>

- A BB is a perfect emitter with unity emissivity ($\varepsilon = 1$).
- All practical bodies have $\varepsilon_\lambda < 1$. The curve below shows the difference between a BB and a non-ideal source.
- The emissivity for a non-ideal source can be obtained through the following experiment.
- Consider an object that is illuminated with incident radiation Φ_o
- From Energy Conservation:

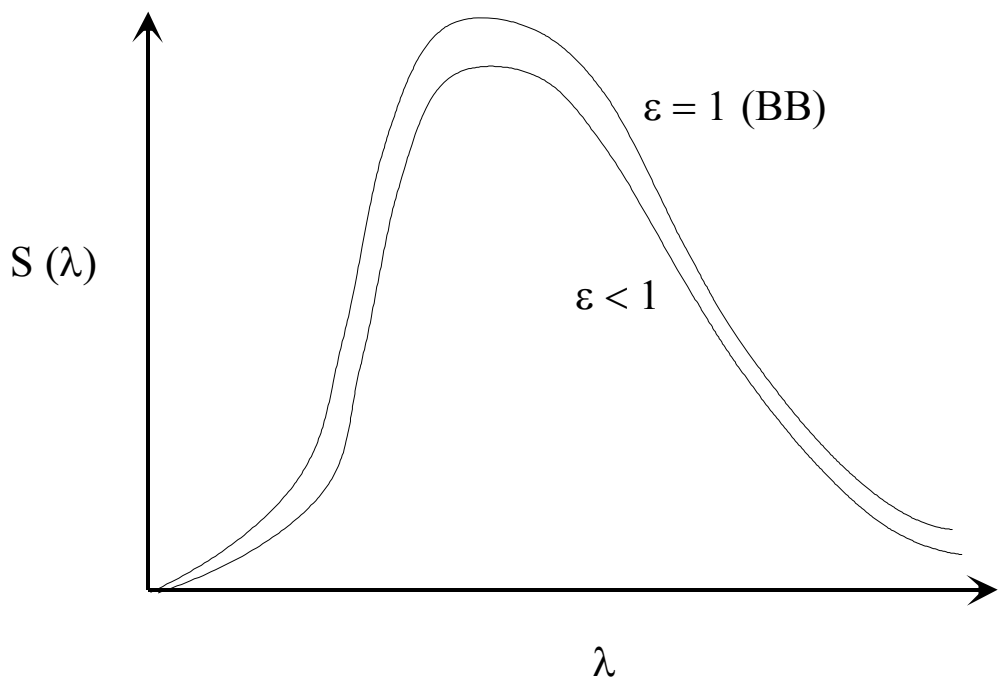
$$R + T_r + \alpha = 1$$

where

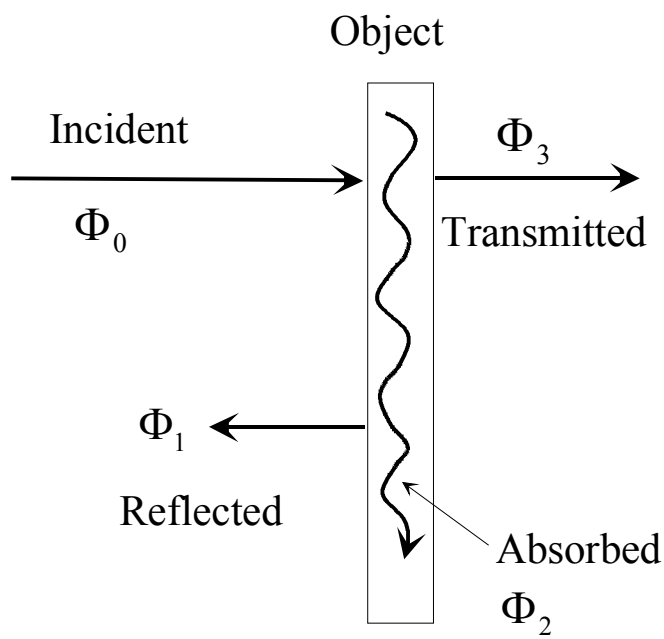
$$R = \frac{\Phi_1}{\Phi_0};$$

$$T_r = \frac{\Phi_3}{\Phi_0};$$

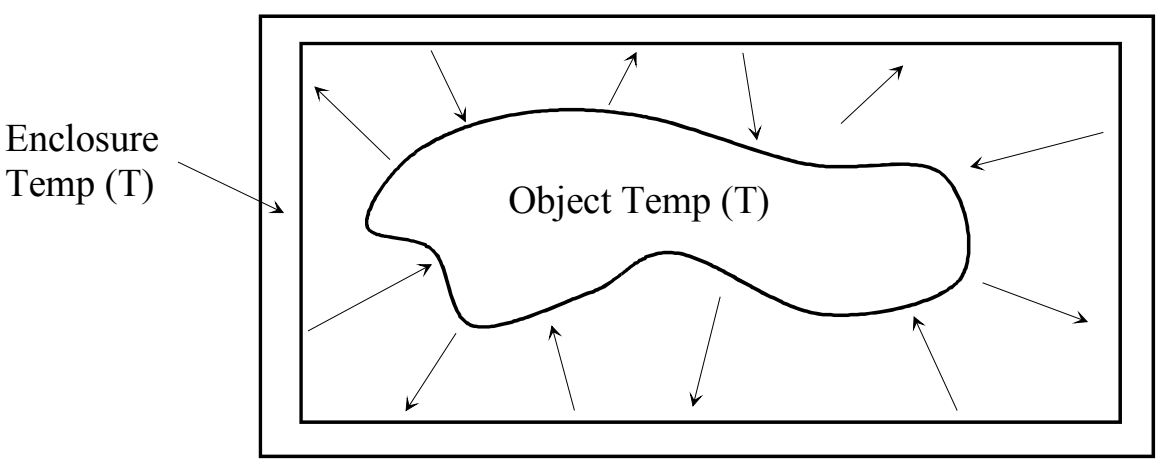
$$\alpha = \frac{\Phi_2}{\Phi_0}$$



The above figure shows the spectrum of a BB and a non-ideal source.



- The emissivity can be shown to be equal to the absorption through the use of the Kirchoff Radiation Law.
- Consider: An object at temperature T surrounded by an enclosure also at temperature T.



- At thermal equilibrium:

$$\Phi_{\substack{Emitted \\ by\ Object}} = \Phi_{\substack{Absorbed \\ by\ Object}}.$$

- If the total flux incident on the object is Φ_0 then the

$$Absorbed\ Flux\ \alpha\Phi_0 = Emitted\ Flux\ \varepsilon\Phi_0$$

and

$$\alpha = \varepsilon \, .$$

Therefore the emissivity coefficient (ε) can be obtained from the absorption coefficient (α) and using

$$R + T_r + \alpha = 1$$

$$\varepsilon = 1 - R - T_r \quad ,$$

and for an opaque object

$$\varepsilon = 1 - R.$$

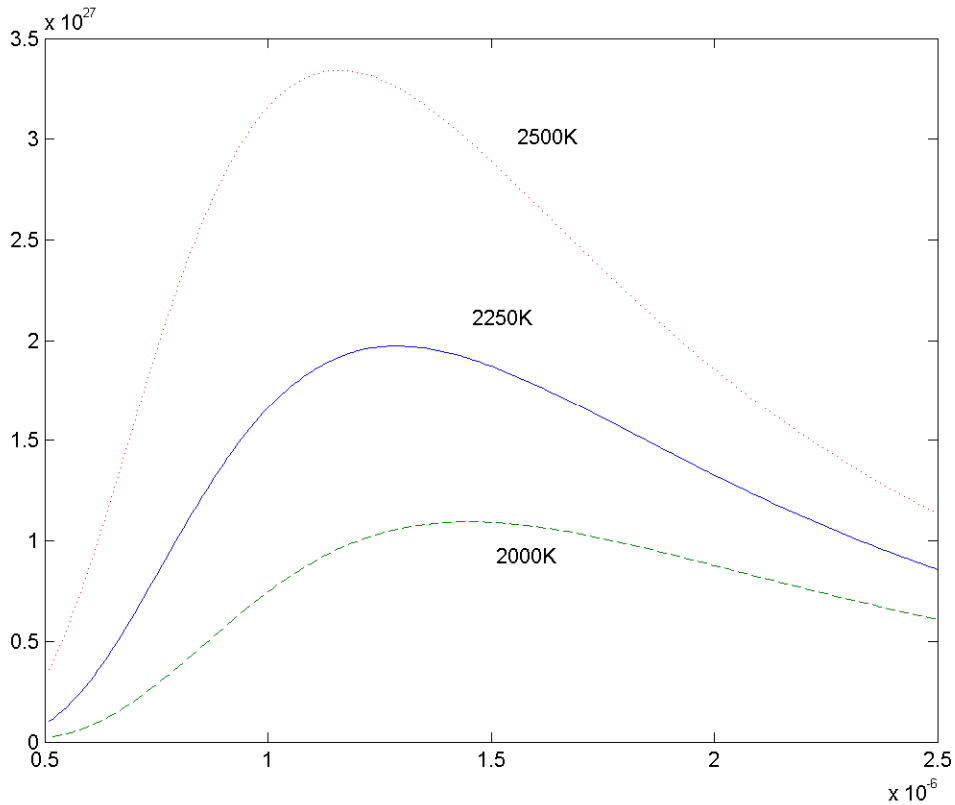
The emissivity will in general be wavelength dependent which would require evaluating the reflectance at each wavelength with a spectrometer. However over a limited spectral range it can be considered a constant.

- Once the emissivity is found the spectral emitance of a non-ideal (BB) source can be found by multiplying M_λ by ε_λ to account for non-ideal emitters.

Wien's Displacement LAW:

The wavelength at which the Planck function is a maximum λ_M times the temperature of the corresponding BB is a constant.

$$\lambda_M T = 2892 \mu m K$$



The figure above shows the relative power emitted from a thermal source at T = 2000°K, 2250°K, and 2500°K as a function of wavelength. The peak wavelength can be predicted using the Wein displacement law.

Example 1: First order radiometric properties of the sun.

Peak solar radiation appears at $\lambda \approx 0.48\mu m$. Therefore

$$T = \frac{2892\mu m - K^o}{0.48\mu m} = 6025^o K \quad .$$

The sun appears as a blackbody source with a temperature near 6000°K.

The peak radiant exitance can be computed:

$$M_{\lambda=0.48\mu m} \approx \frac{3.74\times10^8}{(0.48)^5\left[\exp\left(\frac{1.44\times10^4}{0.48\cdot6025}\right)-1\right]}$$
$$\approx 10^8 W/\left(m^2-\mu m\right)$$

The radiance of the sun at 0.48 μm using $M_\lambda = \pi L_\lambda$:

$$L_{peak} \simeq \frac{10^8}{\pi}\left(W/\left(m^2 sr\mu m\right)\right) \quad \text{(Lambertian Assumption)}$$

The spectral flux from the sun collected on the Earth:

$$\Phi_{\lambda=0.488\mu m} = \int_A\int_\Omega L_{\substack{peak\\sun}} \cos\theta_o d\omega dA$$

$d\omega$ is the angle subtended by the Earth relative to the sun, and dA is an increment of area on the sun's surface. Since the sun is a sphere it appears as a full disk to an observer and the observer appears normal to it. Therefore $\theta_o \approx 0 \rightarrow \cos\theta_o = 1$

Note: Expanding the product of the differentials,

$$dA_{sun}d\omega_{col} = dA_{sun}\frac{dA_{col}}{r^2} = \frac{dA_{sun}}{r^2}dA_{col} = d\omega_{sun}dA_{col}$$

Using a collection area $\Delta A_{col} = 1m^2$ and r = 93×10^6 miles, D_{sun} = 0.865 million miles, then the angle subtended by the sun is

$$\Delta\omega_{sun} = \frac{(0.865/2)^2 \pi}{93^2} \simeq 6.8\times10^{-5} sr \ .$$

The flux from the sun hitting 1 square meter on Earth:

$$\therefore \Phi_\lambda \Big|_{\substack{\lambda=0.48\mu m \\ A=1m^2}} \simeq L_{\substack{sun \\ peak}} \Delta\omega_{sun} \Delta A_{col}$$

$$\simeq \frac{10^8}{\pi}\left(6.5\times10^{-5} sr\right)\left(1m^2\right)$$

$$\simeq 2160 W/m$$

The solar output over all wavelengths:

$$M = \sigma T^4 \simeq 7.35\times10^7 \frac{W}{m^2} \text{ at T = 6000}^{\text{o}}\text{K}$$

$$L = \frac{M}{\pi} \text{ for a Lambertian emitter.}$$

$$\Phi\Big|_{A=1m^2} \simeq \frac{7.35\times10^7}{\pi}\left(6.8\times10^{-5} sr\right)\times 1m^2$$

$$\simeq 1590W$$

for a 1 m^2 collector.

Example 2: Flux collected by a detector on a surface.

$$\Phi_{Collected} = E_i \cos(30^o) A_{Collector} = E_i \cos(30^o)(1cm^2)$$

$$I = \frac{\Phi}{4\pi}, \quad E = \frac{\Phi}{4\pi r^2}, \quad E_i = \frac{I(30^o)}{20cm^2}$$

$$I\left(30^o\right) = I_o \cos\left(30^o\right)$$

$$I_o = L_o A_{source} = \frac{1W}{cm^2 - str} \cdot 1cm^2 = 1\frac{W}{str}$$

$$\therefore \quad E_i = \frac{\left(1W/str\right)\cos(30^o)}{\left(20\,cm\right)^2}$$

$$\Phi_{Collected} = \frac{1 \cdot \cos^2(30^o) \cdot \left(1cm^2\right)}{\left(20\,cm\right)^2}$$

$$= 0.00188W \approx 2\,mW$$