Name Solutions

ECE 340
Fall 2011
Exam 2

November 1, 2011

1. This exam is closed book and closed notes. No talking. No calculators.
2. Useful tables and equations are attached to the end of the test.
3. Show intermediate steps in order to get partial credit if you make an error.

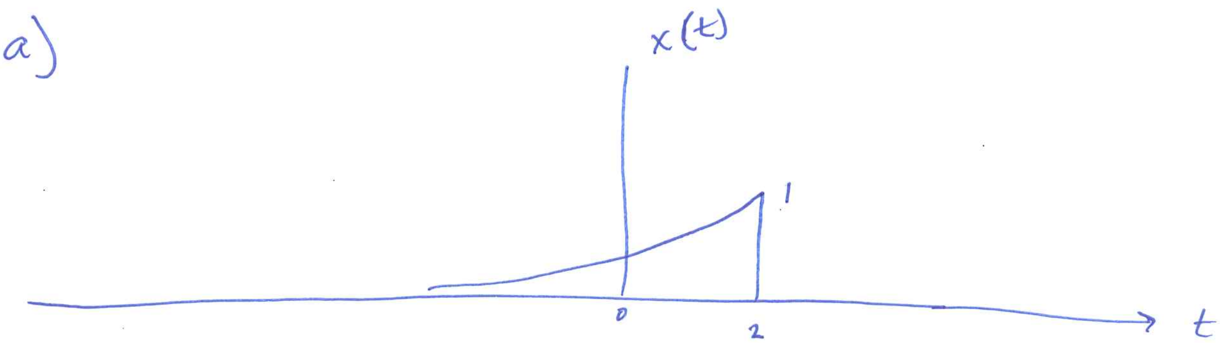
1. (5 Points) A signal is described by $x(t)=e^{(t-2)}u(-t+2)$. Sketch the following:

a) $x(t)$

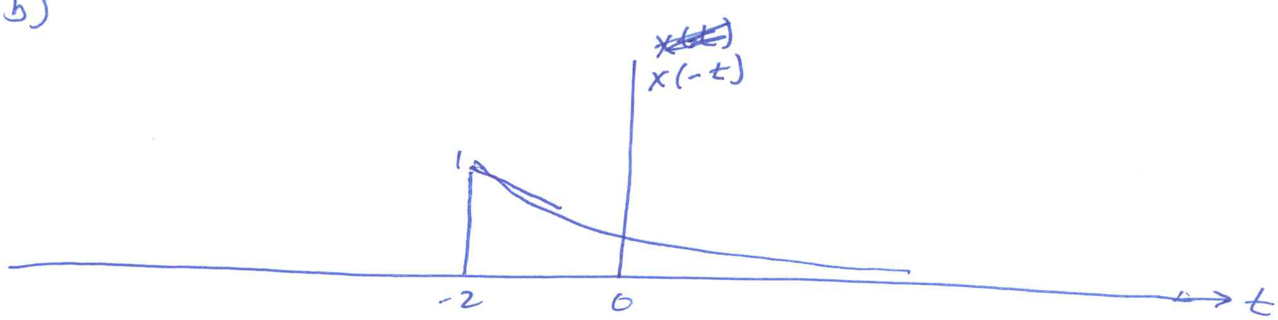
b) $x(-t)$

c) $2x(-t-3)$

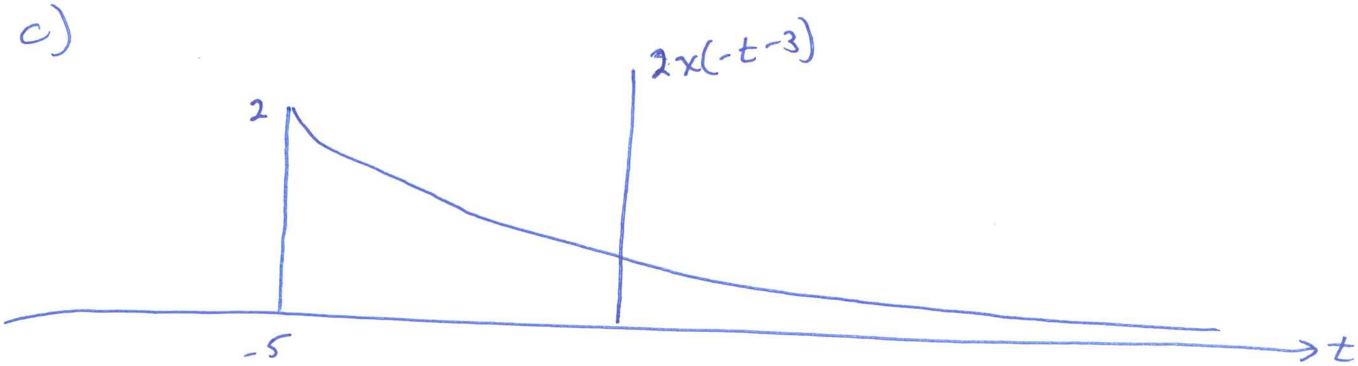
a)



b)



c)



2. (8 Points) For the following statements, answer whether the statement is:
T – always true, P – possibly true, or F – always false.

A. F A real signal having a Fourier transform, will always have an even amplitude and phase spectrum.

B. F The unit step function, $u(t)$, is an energy signal containing ½ joules of energy.

C. F If two sinusoids of different frequencies, f_1 and f_2, are applied at time t = 0 to the input of a linear time-invariant system, the steady-state output of the system will contain sinusoids at frequencies f_1 and f_2, and possibly $f_1 + f_2$ and $f_1 - f_2$.

D. P The Fourier transform of a signal can be obtained by evaluating the signal's Laplace transform on the $s = j\omega$ axis.

–depends on ROC

3. (6 Points)

(a) Write an expression for the Fourier transform of:

$$x(t) = \left[\mathrm{rect}\left(\frac{t}{T}\right) * \Delta\left(\frac{t}{2T}\right)\right]\cos(\omega_0 t).$$

(b) Sketch the phase spectrum for the Fourier transform of:

$$x(t) = \left[\Delta\left(\frac{t-1}{2T}\right) * \Delta\left(\frac{t-2}{2T}\right)\right]\cos(\omega_0 t)$$

a) $x_1(t) = \mathrm{rect}\left(\frac{t}{T}\right) \Leftrightarrow X_1(f) = T\,\mathrm{sinc}\left(\frac{\omega T}{2}\right)$

$x_2(t) = \Delta\left(\frac{t}{2T}\right) \Leftrightarrow X_2(f) = T\,\mathrm{sinc}^2\left(\frac{\omega T}{2}\right)$

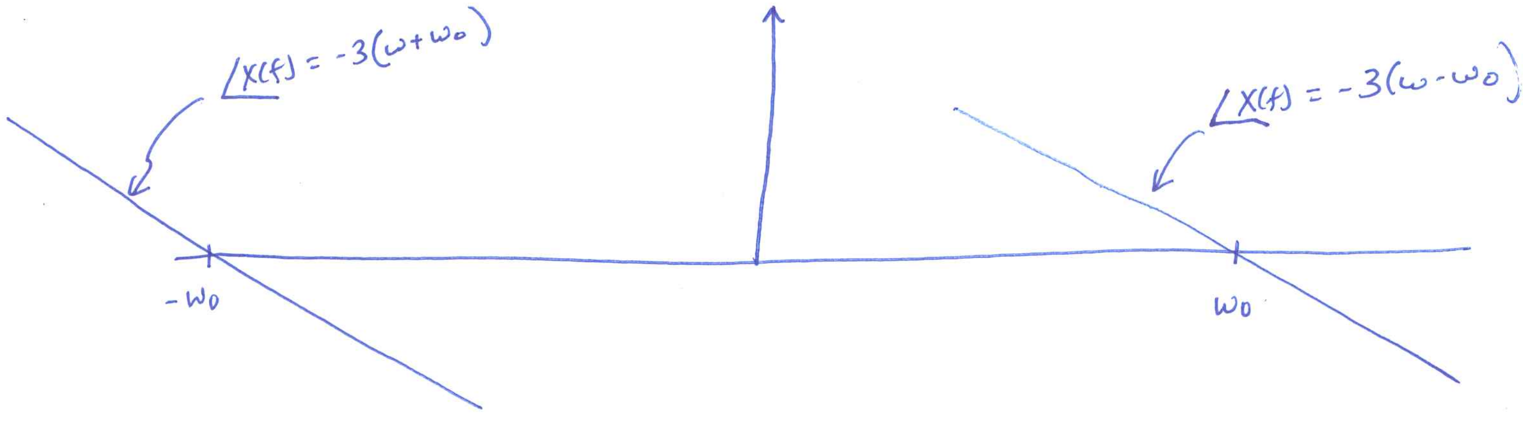
$\Rightarrow \mathcal{F}[x_1(t) * x_2(t)] = X_1(f)X_2(f) = T^2\,\mathrm{sinc}^3\left(\frac{\omega T}{2}\right)$

so $X(f) = \frac{T^2}{2}\left[\mathrm{sinc}^3\left(\frac{(\omega-\omega_0)T}{2}\right) + \mathrm{sinc}^3\left(\frac{(\omega+\omega_0)T}{2}\right)\right]$

b) $\mathcal{F}\left[\Delta\left(\frac{t-1}{2T}\right) * \Delta\left(\frac{t-2}{2T}\right)\right] = T^2\,\mathrm{sinc}^4\left(\frac{\omega T}{2}\right)e^{-j3\omega}$

$\therefore \mathcal{F}[(\Delta(\cdot) * \Delta(\cdot))\cos\omega_0 t] = \frac{T^2}{2}\left[\mathrm{sinc}^4\left(\frac{(\omega-\omega_0)T}{2}\right)e^{-j3(\omega-\omega_0)}\right.$

$\left. + \mathrm{sinc}^4\left(\frac{(\omega+\omega_0)T}{2}\right)e^{-j3(\omega+\omega_0)}\right]$



4. (6 Points) Find the impulse response for the system with input-output relationship described by

$$2\frac{dy(t)}{dt}+y(t)=x(t).$$

$$2s\,Y(s) + Y(s) = X(s)$$

$$= Y(s)\,(1+2s)$$

$$\Rightarrow \frac{Y(s)}{X(s)} = H(s) = \frac{1}{1+2s} = \frac{1/2}{s+1/2} = \frac{1/2}{s-(-1/2)}$$

$$\mathcal{L}^{-1}[H(s)] = \frac{1}{2}e^{-\frac{1}{2}t}u(t) = h(t)$$

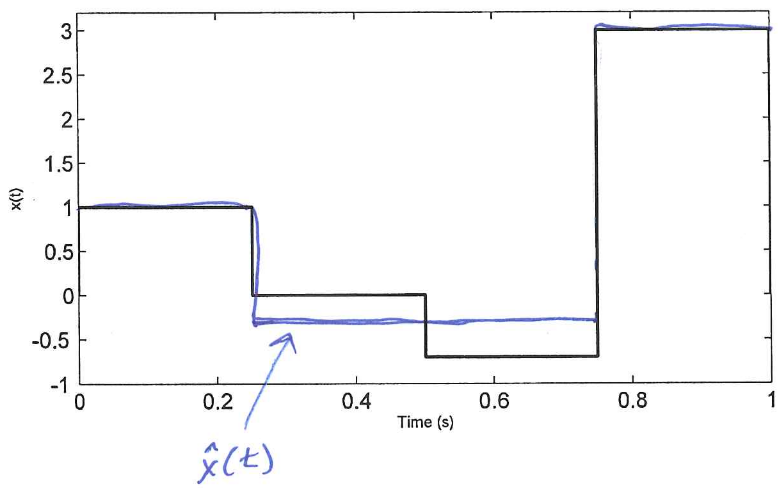
Also,

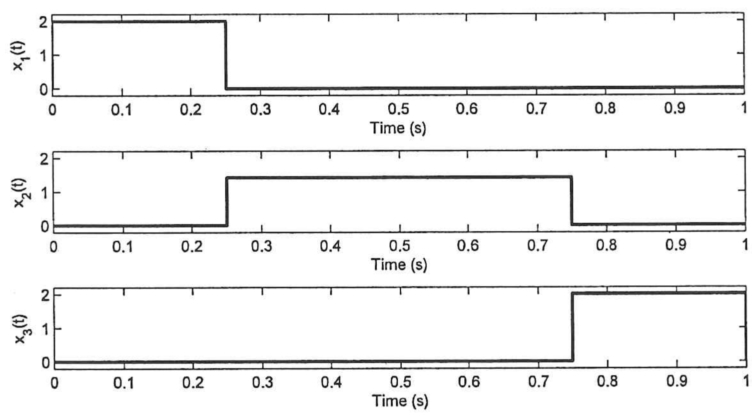
$$Y(s) = \frac{X(s)}{1+2s}, \quad \text{but } X(s) = \mathcal{L}[\delta(t)] = 1$$

$$\therefore\ Y(s) = \frac{1}{1+2s} = \frac{1/2}{\cancel{4}s+1/2}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{2}e^{-\frac{1}{2}t}u(t) = h(t)$$

5. (7 Points). Find the coefficients for the best representation of $x(t)$ below in terms of the three basis functions $x_1(t)$, $x_2(t)$, and $x_3(t)$ (also shown).





$$E_1 = \int_0^{1/4} (2)^2 dt = 1$$

$$E_2 = \int_{1/4}^{3/4} (\sqrt{2})^2 dt = 1$$

$$E_3 = \int_{3/4}^{1} (2)^2 dt = 1$$

$$x(t) \approx a_1 x_1(t) + a_2 x_2(t) + a_3 x_3(t) = \hat{x}(t)$$

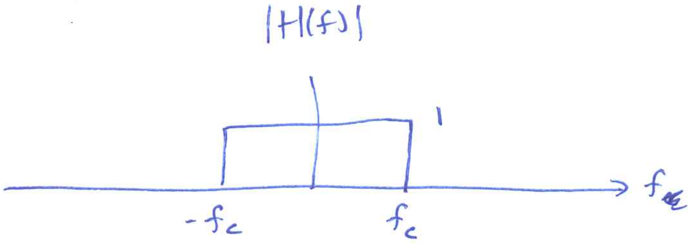
$$a_1 = \frac{1}{E_1}\int_0^1 x(t)\,x_1(t)\,dt = \int_0^{1/4} (2)(1)\,dt = \underline{1/2}$$

$$a_2 = \frac{1}{E_2}\int_0^1 x(t)\,x_2(t)\,dt = \int_{1/4}^{1/2} (\sqrt{2})(0)\,dt + \int_{1/2}^{3/4} (\sqrt{2})\left(\frac{-1}{\sqrt{2}}\right)dt$$

$$= 0 + \left(-\frac{1}{4}\right) = \underline{-1/4}$$

$$a_3 = \frac{1}{E_3}\int_0^1 x(t)\,x_3(t)\,dt = \int_{3/4}^{1} (2)(3)\,dt = \underline{1.5}$$

6. (10 Points) An exponential signal of the form $x(t)=2e^{-t/5}u(t)$ is applied to the input of an *ideal* lowpass filter. Determine the filter cutoff frequency f_c such that half the energy of the input signal will appear at the filter output.



$$x(t) = 2e^{-t/5}u(t) \Leftrightarrow X(f) = \frac{2}{\frac{1}{5}+j2\pi f}$$

$$|X(f)| = \frac{2}{\sqrt{(\frac{1}{5})^2+(2\pi f)^2}}$$

$$|X(f)|^2 = \frac{4}{(\frac{1}{5})^2+(2\pi f)^2}$$

Input Energy:

$$E_x = \int_{-\infty}^{\infty} |X(f)|^2 df = 2\int_0^{\infty} \frac{4}{(\frac{1}{5})^2+(2\pi f)^2} df$$

Using integrals given: $E_x = 10$

Output Energy:

$$E_y = \int_{-f_c}^{f_c} |Y(f)|^2 df = \int_{-f_c}^{f_c} |X(f)|^2 df = 2\int_0^{f_c} \frac{4}{(\frac{1}{5})^2+(2\pi f)^2} df = \frac{1}{2}E_x$$

Using integrals:

$$\frac{20}{\pi}\tan^{-1}(10\pi f_c) = \frac{1}{2}(10) = 5$$

$$\tan^{-1}(10\pi f_c) = \frac{\pi}{4}$$

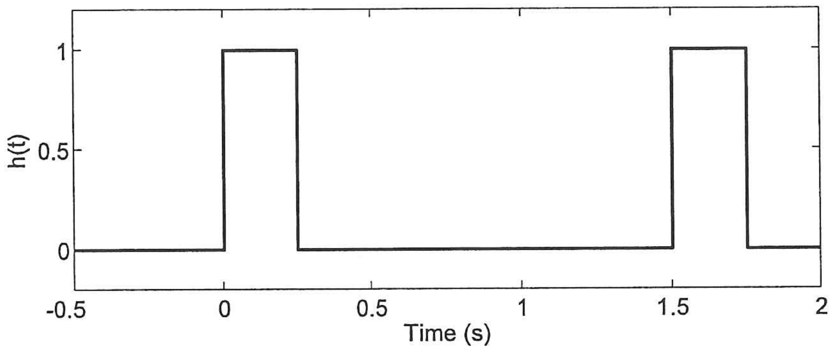
$$\therefore 10\pi f_c = \tan\left(\frac{\pi}{4}\right) = 1$$

$$f_c = \frac{1}{10\pi}\ Hz$$

or $\omega_c = \frac{1}{5}$ rad/s

7. (8 Points) A LTIC system has the impulse response shown below. What is the steady-state output of this system due to an input signal given by:

$$x(t) = \cos^2(\pi t) u(t).$$



$x(t) = \cos^2(\pi t) u(t) = \cancel{\tfrac{1}{2}\cos}$

$= \cos(\pi t)\cos(\pi t) u(t) = \frac{1}{2}\left[\cos(0) + \cos(2\pi t)\right] u(t)$

$= \left[\frac{1}{2} + \frac{1}{2}\cos(2\pi t)\right] u(t) = \left[\frac{1}{2} + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right)\right] \cancel{dt}\, u(t)$

For: steady state: $e^{j\omega t} \to \boxed{H(\omega)} \to H(\omega) e^{j\omega t}$

$H(\omega) = \mathcal{F}\left[\mathrm{rect}\left(\frac{t - 1/8}{1/4}\right) + \mathrm{rect}\left(\frac{t - 13/8}{1/4}\right)\right]$

$= \frac{1}{4}\,\mathrm{sinc}\left(\frac{\omega}{8}\right) e^{-j\omega(1/8)} + \frac{1}{4}\,\mathrm{sinc}\left(\frac{\omega}{8}\right) e^{-j\omega(13/8)}$

$H(\omega = 0) = \frac{1}{4} + \frac{1}{4} = 1/2$

$H(\omega = 2\pi) = \frac{1}{4}\,\mathrm{sinc}(\pi/4)\left[e^{-j\pi/4} + e^{-j13\pi/4}\right]$

$= \frac{1}{4}\,\frac{\sin(\pi/4)}{\pi/4}\left[\frac{1}{\sqrt{2}}(1 - j) + \frac{1}{\sqrt{2}}(-1 + j)\right]$

$= 0$

by symmetry $H(\omega = -2\pi) = 0$

$\therefore\ y_{ss}(t) = H(\omega = 0) \cdot \frac{1}{2} e^{j(0)t} = \frac{1}{2}\left(\frac{1}{2}\right) = \boxed{\frac{1}{4}}$