2-D FILTER DESIGN

TYPES

- Finite Impulse Response (FIR) described here
- Infinite Impulse Response (IIR) not discussed
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ZERO-PHASE FILTERS

- $h(m,n)$ is zero-phase if $H(k, l) = H^*(k, l)$, i.e. $H$ is real

- Zero-phase desired in digital filters to avoid signal distortion

Zero-phase not guaranteed in physical filters, e.g. optical defocus and square scan spots (see Notes 4 and 5)

- $H$ real implies

  $$h(m, n) = h^*(-m, -n)$$

  Prove the above equation
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- Furthermore, since $h$ is assumed real in our case,

\[ h(m, n) = h(-m, -n), \text{ i.e. } h \text{ is two-fold symmetric} \]

- Greater symmetry reduces number of independent points (Degrees of Freedom (DOF)) required in filter design
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- Additional symmetries

**four-fold (includes two-fold):** \( h(m, n) = h(-m, n) = h(m, -n) \)

\[ \begin{array}{c}
\text{four-fold (includes two-fold): } h(m, n) = h(-m, n) = h(m, -n) \\
\end{array} \]

**eight-fold (includes four-fold):** \( h(m, n) = h(-m, n) = h(m, -n) = h(n, m) \)

\[ \begin{array}{c}
\text{eight-fold (includes four-fold): } h(m, n) = h(-m, n) = h(m, -n) = h(n, m) \\
\end{array} \]
# 2-D Filter Design

## Filter Specification

- **Frequency domain**

- **Ideal, continuous, radial filters (note, none are FIR)**

<table>
<thead>
<tr>
<th>type</th>
<th>( F(\rho) )</th>
<th>( f(r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Low-Pass Filter (LPF)</strong></td>
<td>( \text{cyl} \left( \frac{\rho}{2\rho_c} \right) )</td>
<td>( \frac{\pi}{4} \text{somb}(2\rho_c r) )</td>
</tr>
<tr>
<td><strong>High-Pass Filter (HPF)</strong></td>
<td>( 1 - \text{cyl} \left( \frac{\rho}{2\rho_c} \right) )</td>
<td>( \frac{\delta(r)}{\pi r} - \frac{\pi}{4} \text{somb}(2\rho_c r) )</td>
</tr>
<tr>
<td><strong>Band-Pass Filter (BPF)</strong></td>
<td>( \text{cyl} \left( \frac{\rho}{2\rho_2} \right) - \text{cyl} \left( \frac{\rho}{2\rho_1} \right), \rho_2 &gt; \rho_1 )</td>
<td>( \frac{\pi}{4} \left[ \text{somb}(2\rho_2 r_2) - \text{somb}(2\rho_1 r_1) \right] )</td>
</tr>
<tr>
<td><strong>Band-Stop Filter (BSF)</strong></td>
<td>( 1 - \text{cyl} \left( \frac{\rho}{2\rho_2} \right) + \text{cyl} \left( \frac{\rho}{2\rho_1} \right), \rho_2 &gt; \rho_1 )</td>
<td>( \frac{\delta(r)}{\pi r} - \frac{\pi}{4} \left[ \text{somb}(2\rho_2 r_2) - \text{somb}(2\rho_2 r_2) \right] )</td>
</tr>
</tbody>
</table>
2-D FILTER DESIGN

- Amplitude transition cannot be sharp with real filters

- Therefore, specify transition region and tolerances in designed filters

in passband region: \(1 - \varepsilon_p < |H(u, v)| < 1 + \varepsilon_p\)

in stopband region: \(|H(u, v)| < \varepsilon_s\)

which measure quality of filter relative to ideal case
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FILTER DESIGN

• Assume a continuous, periodic frequency domain filter and a discrete spatial domain filter, with the Discrete-Space Fourier Transform relation,

\[
H(u, v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h(m, n) e^{-j2\pi um} e^{-j2\pi vn}
\]

\[
h(m, n) = \int_{u=-1/2}^{1/2} \int_{v=-1/2}^{1/2} H(u, v) e^{j2\pi um} e^{j2\pi vn} dudv
\]

Rewrite in terms of angular frequencies \(\omega_1 = 2\pi u, \omega_2 = 2\pi v\)

Examples

• Specify \(H\) and calculate \(h\)

\[
H(u, v) = \text{rect}\left(\frac{u}{a}, \frac{v}{b}\right), \quad |u| \leq 1/2, \quad |v| < 1/2
\]

\[
h(m, n) = |a||b|\text{sinc}(am, bn)
\]
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- Specify \( h \) and calculate \( H \)

\[
H(u, v) = \sum_{m = -\infty}^{\infty} \sum_{n = -\infty}^{\infty} h(m, n)e^{-j2\pi um}e^{-j2\pi vn} \\
= \frac{1}{3} + \frac{1}{6}(e^{-ju} + e^{ju} + e^{-jv} + e^{jv}) \\
= \frac{1}{3} + \frac{1}{6}(2\cos u + 2\cos v) \\
= \frac{1}{3}(1 + \cos u + \cos v)
\]
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Methods for Filter Design

- window (1-D extension)
- frequency sampling (1-D extension)
- transformation (unique to 2-D)
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Window Method

- Assume a desired frequency response $H_d(u, v)$

- Ideal frequency domain filters result in Infinite Impulse Responses (IIR)

- Limit the extent of the spatial response with a window function $w$

  $$h(m, n) = h_d(m, n)w(m, n)$$

- Corresponds to a convolution in frequency domain

  $$H(u, v) = H_d(u, v) \ast \ast W(u, v)$$

- Desirable properties of $W(u, v)$:
  - narrow main-lobe
  - low side-lobes
  - finite extent in spatial domain
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Generation of 2-D windows from 1-D windows

- Two ways:

  Implement 1-D window functions in separable 2-D window

  \[ w(m, n) = w_1(m)w_2(n) \]
  \[ W(u, v) = W_1(u)W_2(v) \]

  Rotate 1-D window about the origin as “generating function” to create 2-D radial (circularly symmetric) function (see Notes2)

  \[ w(m, n) = w(\sqrt{m^2 + n^2}) \]
  \[ W(u, v) = W(\sqrt{u^2 + v^2}) \]
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- Examples

- All defined with:

  width $b=8$

  plot array $M=N=32$

  spatial region of support = 8 x 8 (separable) or radius = 4 (radial)

  plot layout:

```
\[
\begin{array}{c}
\begin{array}{c}
\text{separable} \\
\text{version}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
w(x,y) \\
|W(u,v)|
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
w(r) \\
|W(\rho)|
\end{array}
\end{array}
\begin{array}{c}
\text{radial} \\
\text{version}
\end{array}
\end{array}
\]```
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- rectangle \( w(x) = 1 \) (see Notes1)

- cylinder \( w(r) = cyl\left(\frac{r}{b}\right) \)

\[ \text{rectangle window - separable version} \quad \text{radial version} \]
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- **Hanning (raised cosine)**  
  \[ w(r) = 0.5 + 0.5 \cos \left( 2\pi \frac{r}{b} \right) \]
• **Hamming** \( w(r) = 0.54 + 0.46 \cos \left( 2\pi \frac{r}{b} \right) \)
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- **Kaiser** \( w(r) = \frac{1}{I_0(\alpha)} I_0(\alpha \sqrt{1 - (2r/b)^2}) \)

where \( I_0 \) is a modified Bessel function of the first kind, order zero, and \( \alpha \) controls the window shape:
  - \( \alpha \) smaller \( \Rightarrow \) mainlobe narrower, \( \alpha \) larger \( \Rightarrow \) sidelobes lower

\( \alpha = 1 \) radial version \hspace{1cm} \( \alpha = 3 \)
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Frequency Sampling Method

- **Sample 2-D continuous frequency domain filter** $H(u,v)$ to obtain $H(k,l)$ (origin-centered)

- **Calculate inverse DFT to obtain** $h(m,n)$ (origin-centered)

**Frequency Sampling Method**

- **add linear phase (M and N odd)**
  
  $H_1(u, v) = H_z(u, v)e^{-j2\pi u((M-1)/2)}e^{-j2\pi v((N-1)/2)}$

  $H_1(k, l) = H_1(k/M, l/N)$

- **sample**
  
  $k = 0, 1, \ldots M - 1$
  
  $l = 0, 1, \ldots N - 1$

- **inverse DFT** $h_1(m, n)$

- **shift** $h_z(m, n) = h_1(m + (M - 1)/2, n + (N - 1)/2)$
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- Sampling of ideal frequency domain filters results in discontinuities at cut-off and cut-on frequencies

- Apply “smoothing” in transition region

  Simple linear transition works fine

- Origin-centering important to maintain zero phase