CONVOLUTION VIA FOURIER TRANSFORMS

- **Convolution Theorem**

  \[
  \text{Fourier transform of } f \text{ is } \hat{f} \text{, so must zero-pad } h \text{ to same size as } f \text{, so that size}(H) = \text{size}(F)
  \]

  Usually size(h) > size(f), so must zero-pad h to same size as f, so that size(h) = size(f)

  To do convolution using Fourier transforms,

  1. Transform f and h to obtain F and H
  2. Multiply F and H to obtain G
  3. Inverse transform G to obtain g

  Must have same number of points in each array F and H in order to multiply.

  \[
  \text{gf} \leftrightarrow \text{GH} \\
  \text{Convolution Theorem}
  \]

**IMAGE ENHANCEMENT III (Fourier)**)
Note, multiplication in Fourier domain is full complex operation.
Zero-pad $h$ to match size of $f$.

Examples with box filter:

- $h: 8 \times 8 \rightarrow 256 \times 256$
- $h: 32 \times 32 \rightarrow 256 \times 256$

Results in circular convolution.

Zero-pad $h$ to match size of $f$. (Fourier Image Enhancement III)
Phase Effect of Padding

Example with:

- Reference cross is at (128, 128)
- Size (filter) = K x K = 16 x 16
- Size (image) = M x M = 256 x 256

Reveals periodic extension
Zero-pad h and f

• Extract center M x N

• Radix-2 FFT requires power of 2

\[ p \times q = (M + L - 1) \times (N + K - 1) \]

• Radix-2 FFT requires power of 2

• Results in linear convolution

For example, if \( M = N = 512 \) and \( K = L = 16 \), then \( p = q = 1024 \)

• Pad both to at least

IMAGE ENHANCEMENT III (Fourier)
Note, border artifact due to linear convolution with zero padding.
• Fill image array with mean DN to reduce (but not eliminate) border artifact.
Overlap-Add Method

- Zero-padding h and f wastes memory and computation
- Overlap-add partitions image into $I \times J$ smaller blocks, pads each block and filters h to same size, filters each pads each block and filter h to same size, filters each
- Overlap-add partitions image into $I \times J$ smaller blocks, overlaps blocks separately, and recombines
- Resources
  - Zero-padding h and f wastes memory and computation

where the convolution is linear

$$
[(u, w)_h \ast \ast (u, w)_f] \sum_{f} \sum_{i} = 
$$

$$
(u, w)_h \ast \ast \left[ (u, w)_f \sum_{f} \sum_{i} \right] = (u, w)_h \ast \ast (u, w)_f
$$

$$
(u, w)_f \sum_{f} \sum_{i} = (u, w)_f
$$

**Image Enhancement III (Fourier)**
Example: Image Enhancement III (Fourier)

- Extract center 512x512 image
- Overlap (by 118x118) and add 128x128 blocks
- Convolution of each block
- Filter each block using DFT (results in linear convolution of each block)
- Pad blocks and h to 128x128
- Partition image into blocks, each 118x118, plus smaller blocks along 2 edges
- Image M=N=512, Filter K=L=118

- Example overlap-add of 4 128x128 blocks
- Partitioning of image into 25 blocks
- 128x128 blocks
- 118x118 blocks
- 40x128
**TRANSFER FUNCTION SPECIFICATION (Fourier)**

- Only 2 DFTs required: Image and inverse of Image-Filter product
- Equivalent to convolution, circular or linear, depending on padding
- Only 2 DFTs required: Image and inverse of Image-Filter product

**IMAGE ENHANCEMENT III (Fourier)**
Example with Gaussian LP transfer function, $r_e = 1/e$ radius, i.e. where $H(r_e) = H(0)/e$

- $r_e = 1/8$ cycles/pixel
- $r_e = 1/4$ cycles/pixel
- $r_e = 1/16$ cycles/pixel
**IMAGE ENHANCEMENT III (Fourier)**

**SPATIAL PHASE COMPONENT**

- Carries image gradient information
- Techniques (later)
- Phase-only reconstruction like a HP-filtered component

\[ \text{amp}(F) \quad \text{DFT}^{-1} [\text{amp}(F)] \]

\[ \text{pha}(F) \quad \text{DFT}^{-1} [\text{pha}(F)] \]