THE DISCRETE FOURIER TRANSFORM

Definition

Relation to the continuous Fourier transform

Calculation

Application
**DEFINITION**

**forward DFT**

\[
F_{uv} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f_{xy} \cdot e^{-j2\pi(ux/M + vy/N)}
\]

**inverse DFT**

\[
f_{xy} = \frac{1}{MN} \cdot \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F_{uv} \cdot e^{j2\pi(ux/M + vy/N)}
\]

Maps a **discrete, complex 2-D array of size** \( M \times N \) **into another discrete, complex 2-D array of size** \( M \times N \).

Approximates the **Continuous Fourier Transform (CFT)** **under certain conditions.**
RELATION OF THE DFT TO THE CFT

One model for the DFT as an approximation to the CFT (aka “a recipe”):

- 1. sample \( f(x,y) \) at intervals of \( X \) and \( Y \)

\[
f(x, y) \cdot \frac{1}{XY} \text{comb}(x/X, y/Y)
\]

- 2. truncate to \( MX \times NY \)

\[
f(x, y) \cdot \frac{1}{XY} \text{comb}(x/X, y/Y) \cdot \text{rect}(x/MX, y/NY)
\]

- 3. make periodic

\[
f(x, y) \cdot \frac{1}{XY} \text{comb}(x/X, y/Y) \cdot \text{rect}(x/MX, y/NY)
\]

\[\otimes \frac{1}{MX \cdot NY} \text{comb}(x/MX, y/NY)\]
4. take CFT

\[ F(u, v) \otimes \text{comb}(uX, vY) \otimes MX \cdot NY \text{sinc}(uMX, vNY) \cdot \text{comb}(uMX, vNY) \]

- replicate
- smooth
- sample
CALCULATION

Both functions, $f_{xy}$ and $F_{uv}$, are

- periodic (period = $M \times N$ points or $MX \times NY$ units) and

- sampled ($X \times Y$ units in space, $1/MX \times 1/NY$ in frequency)

If one function has compact support (i.e. it is space- or frequency-limited), the other must have $\infty$ support

- Therefore, aliasing will occur with the DFT, either in space or frequency.

- To approximate the CFT by the DFT, aliasing must be minimized in both domains.

The Fast Fourier Transform (FFT) is an efficient algorithm to calculate the DFT
ARRAY COORDINATES

Because of the array-oriented coordinate system, the output of a DFT (or FFT) program is not “natural.” The \( u = 0 \) and \( v = 0 \) coordinates are column 0 and row 0 in the raw format.

\[
\begin{array}{c|c|c|c}
\text{I} & II & \\ 
III & IV & \\ 
IV & III & \\ 
\end{array}
\]

\[
\begin{array}{c|c|c|c}
I & II & \\ 
III & IV & \\ 
II & I & \\ 
\end{array}
\]

\( N \) and \( M \) are commonly powers of 2 (for the FFT). Therefore, the \( u = 0 \) and \( v = 0 \) coordinates are at column \( N/2 \) and row \( M/2 \) in the reordered format.
SAMPLE INTERVALS

Constraints

- **product of sample intervals in x and u:** \[ XU = 1/M \]
  
  **in y and v:** \[ YV = 1/N \]

- **replication frequency in u:** \[ u_r = 1/X \]
  
  **in v:** \[ v_r = 1/Y \]

- **folding frequency in u:** \[ u_f = 1/2X \]
  
  **in v:** \[ v_f = 1/2Y \]

*For digital images, a normalized coordinate system is*
• $X = 1 \text{ pixel}$

• $Y = 1 \text{ pixel}$

*Therefore,*

• $U = 1/M \text{ cycles/pixel, } u_r = 1 \text{ cycle/pixel, } u_f = 1/2 \text{ cycle/pixel}$

• $V = 1/N \text{ cycles/pixel, } v_r = 1 \text{ cycle/pixel, } v_f = 1/2 \text{ cycle/pixel}$
EXAMPLES

Reordering DFT

\( f_{xy} \)

\(|F_{uv}|\)

reordered \(|F_{uv}|\)
format
Aliasing in frequency domain

3 x 3

5 x 5

9 x 9

\[ f_{xy} \]

\[ |F_{uv}| \]

\[ |F_{uv}| - |F(u,v)| \]