ECE 425

Image Science and Engineering

Spring Semester 2000

Course Notes

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ECE 402
DEFINITIONS
Image science

- The theory of optical image formation and detection

- Includes elements of:
  - optics
  - radiometry
  - linear systems
  - statistics
  - vision
Image engineering

- The technologies of image acquisition, transmission, storage and display

- Includes elements of:
  - detectors
  - signal processing
  - data compression
  - image processing
OVERVIEW

Electronic imaging systems pervade modern life

Examples

- **Document processing** (scanning, storage, printing, digital libraries, WWW)

- **Consumer products** (HDTV, digital cameras, photo scanners and printers)

- **Machine vision** (quality control inspection, robotics)

- **Medical imaging** (disease diagnosis, medication monitoring)

- **Scientific visualization** (complex mathematical models, interactive graphics)
- Remote sensing (earth science, environmental monitoring, weather)

- Military (reconnaissance, surveillance, targeting)
THE SYSTEMS APPROACH

A perceived image is the result of a chain of systems:

- optics
- detector
- coding/decoding
- display
- human vision

Each can be considered a subsystem of the total electronic imaging system

The engineering design goal is to optimize the
performance of each subsystem in relation to that of the others and the total system

This course covers the tools used for imaging system analysis, design and evaluation
An imaging system consists of several subsystems

- *points of signal transduction, optical $\leftrightarrow$ electronic*
For optical components, we’re concerned with:

- size and location of the image (geometrical optics)
- intensity of the image (radiometry)
- contrast and sharpness of the image (linear systems)
For electronic components, we’re concerned with:

- image sampling and quantization (analog filters, A/D converters, coding)

- image processing (digital signal processing)
SECTION I - MATHEMATICAL TOOLS

Mathematics Background

Convolution and Fourier Transforms

Linear Filtering and Sampling

Two-dimensional Functions and Operations

Discrete Fourier Transform and Fast Fourier Transform
MATHEMATICS BACKGROUND
Complex Notation

- Complex arithmetic will be necessary for Fourier analysis and optics

- Complex numbers consist of two real numbers, joined by a phasor relationship

\[ c = a + jb \]

- where
  - \( c \) is a complex number,
  - \( a \) is the real part of \( c \)
  - \( b \) is the imaginary part of \( c \)
  - \( j = \sqrt{-1} \)

- Phasor relationship
- **The amplitude** $A$ of $c$
  \[ A = \sqrt{a^2 + b^2} \]

- **The phase** $\theta$ of $c$
  \[ \theta = \tan^{-1}(b/a) \]

- Can write $c$ as $c = Ae^{j\theta}$ which, by Euler’s Theorem,
  \[
  c = A(\cos \theta + j\sin \theta)
  = A\left(\frac{a}{A} + j\frac{b}{A}\right)
  = a + jb
  \]
Using Euler’s Theorem, we can derive the following relations:

\[ \sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta}) \]
\[ \cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \]

Using Euler’s Theorem, show that

\[ (\sin\theta)^2 + (\cos\theta)^2 = 1 \]
Simple Functions

- **Delta function and its relatives**
  - **delta** \( \delta(x - x_0) \)

  \[ \delta(x - x_0) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases} \]

  \[ \delta(\frac{x - x_0}{b}) = |b|[\delta(x - x_0 + b) + \delta(x - x_0 - b)] \]

- **NOTE:** The delta function’s amplitude is infinite and its area is 1. The amplitude is shown as 1 for convenience in plots.

  - Write the equation that defines the area of a delta function as 1.
  - Review the definition of delta function in terms of the limit of conventional functions, such as the rectangle function

- **even delta pair**

  \[ \delta \left( \frac{x - x_0}{b} \right) = |b|[\delta(x - x_0 + b) + \delta(x - x_0 - b)] \]
- **odd delta pair**

\[ \delta \delta \left( \frac{x - x_0}{b} \right) = |b| [\delta(x - x_0 + b) - \delta(x - x_0 - b)] \]
• **comb (shah)**

\[
comb\left(\frac{x-x_0}{b}\right) = |b| \sum_{n=-\infty}^{\infty} \delta(x-x_0-nb)
\]

- **x₀ = 0**

- **x₀ ≠ 0**

Even delta pair, odd delta pair and comb functions are all scaled by b
**Use of the δ function**

- **sifting**
  \[ \int_{-\infty}^{\infty} f(\alpha)\delta(\alpha - x_0)d\alpha = f(x_0) = \text{constant} \]

- **NOTE:** Sifting is a convolution, evaluated for a particular shift

- **sampling**
  \[ f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0) \]
- **NOTE:** Sampling is a multiplication

- Output is a delta function, with area determined by the value of the function at the specified value of the independent variable.

**uniform sampling**

\[
\frac{1}{|b|} f(x) \text{comb}\left(\frac{x-x_0}{b}\right) = \sum_{n=-\infty}^{\infty} f(x_0 + nb) \delta(x - x_0 - nb)
\]

[Diagram of uniform sampling]

\[\text{x} \quad f(x) \quad = \quad f(x_0) \quad \text{x} \]

\[\text{x}_0 \quad \text{x} \]

\[\text{x}_0 \quad \text{x} \]

\[\text{x}_0 \quad \text{x} \]
- **NOTE:** Must divide comb function by |b| to retain amplitude of \( f(x) \).

- **NOTE:** \( f(x) \) modulates the comb function.

- **shifting** \( g(x) = f(x) \ast \delta(x-x_0) = \int_{-\infty}^{\infty} f(\alpha) \delta(x-x_0-\alpha) d\alpha = f(x-x_0) \)

- **replicating** \( g(x) = \frac{1}{|b|} f(x) \ast \text{comb} \left( \frac{x-x_0}{b} \right) \)
• **NOTE:** Must divide by $|b|$ to retain amplitude of $f(x)$

• **Other 1-D functions**

  • **rectangle (square pulse)**

    $$\text{rect}\left(\frac{x}{b}\right) = \begin{cases} 
    0 & |x/b| > 1/2 \\
    1/2 & |x/b| = 1/2 \\
    1 & |x/b| < 1/2 
  \end{cases}$$

  • **triangle**

    $$\text{tri}\left(\frac{x}{b}\right) = \begin{cases} 
    0 & |x/b| \geq 1 \\
    1 - |x/b| & |x/b| < 1 
  \end{cases}$$
What is the value of $b$ in the above graph?

- **sinc**
  \[
  \text{sinc}(x/b) = \frac{\sin(\pi x/b)}{\pi x/b}
  \]

- **sinc-squared**
  \[
  \text{sinc}^2(x/b)
  \]

For a given $b$, the tri function is twice as wide as the rect function.
What is the value of $b$ in the above graph?

- $\text{gaus(sian)}$

$$\text{gaus}(x/b) = e^{-\pi(x/b)^2}$$
What is the value of \( b \) in the above graph?

- **cosine**
  \[
  \cos \left( \frac{2\pi x}{b} \right) = \frac{e^{j2\pi(x/b)} + e^{-j2\pi(x/b)}}{2}
  \]

- **sine**
  \[
  \sin \left( \frac{2\pi x}{b} \right) = \frac{e^{j2\pi(x/b)} - e^{-j2\pi(x/b)}}{2j}
  \]
What is the value of $b$ in the above graph?
CONVOLUTION AND FOURIER TRANSFORMS (1-D)

Convolution (1-D)

- **Why is it important?**
  - Describes the effect of a Linear Shift Invariant (LSI) system on input signals

- **L is the system operator**
- **Can write a general description of any system as,**
  \[ g(x) = L[f(x)] \]

- **For an LSI system, L is a convolution,**
  \[ g(x) = f(x) \ast h(x), \text{ of the input signal and the system impulse response, } h(x) \]
More on this later

Mathematical and graphical description

\[ g(x) = f(x) \ast h(x) = \int_{-\infty}^{\infty} f(\alpha)h(x - \alpha)d\alpha \]

Example
Convolution

1. write both as a function of $\alpha$  
   $f(\alpha)$ and $h(\alpha)$

2. flip $h$ (or $f$) about $\alpha = 0$  
   $h(-\alpha)$

3. shift $h$ (or $f$) by an amount $x$  
   $h(x - \alpha)$

4. multiply the two functions  
   $f(\alpha)h(x - \alpha)$

5. integrate the product function over all $\alpha$  
   $g(x)$

6. repeat steps 3 through 5 until done
\[ \alpha \]

\[ h(\alpha) \]

\[ \alpha \]

\[ f(\alpha)h(0 - \alpha) \]

\[ \alpha \]

\[ \text{area} = g(0) \]

\[ f(\alpha)h(1 - \alpha) \]

\[ \alpha \]

\[ \text{area} = g(1) \]

\[ f(\alpha)h(2 - \alpha) \]

\[ \alpha \]

\[ \text{area} = g(2) \]

\[ f(\alpha)h(3 - \alpha) \]

\[ \alpha \]

\[ \text{area} = g(3) \]

\[ f(\alpha)h(4 - \alpha) \]

\[ \alpha \]

\[ \text{area} = g(4) \]
Plot $g(x)$

The shifts in this example are by integer steps, for illustration convenience.
1-D CONVOLUTION PROPERTIES

<table>
<thead>
<tr>
<th>property</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>commutative</td>
<td>( f(x) \ast h(x) = h(x) \ast f(x) )</td>
</tr>
<tr>
<td>distributive</td>
<td>( f(x) \ast [h_1(x) + h_2(x)] = f(x) \ast h_1(x) + f(x) \ast h_2(x) )</td>
</tr>
<tr>
<td>associative</td>
<td>( f(x) \ast [h_1(x) \ast h_2(x)] = [f(x) \ast h_1(x)] \ast h_2(x) )</td>
</tr>
</tbody>
</table>
### 1-D Convolution Examples

<table>
<thead>
<tr>
<th>f(x)</th>
<th>h(x)</th>
<th>g(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>δ(x)</td>
<td>f(x)</td>
</tr>
<tr>
<td>f(x-x₀)</td>
<td>h(x)</td>
<td>g(x-x₀)</td>
</tr>
<tr>
<td>f(x)</td>
<td>h(x-x₀)</td>
<td>g(x-x₀)</td>
</tr>
<tr>
<td>rect(x)</td>
<td>rect(x)</td>
<td>tri(x)</td>
</tr>
<tr>
<td>sinc(x)</td>
<td>sinc(x)</td>
<td>sinc(x)</td>
</tr>
</tbody>
</table>
| gaus(x) | gaus(x) | \( \frac{1}{\sqrt{2}} gaus \left( \frac{x}{\sqrt{2}} \right) \)
Fourier Transforms (1-D)

- Why is it important?

- For an LSI system, the convolution operator becomes a multiplicative operator in the Fourier domain
  - Taking the Fourier transform of the system equation,
\[ G(u) = F(u)H(u) \]

- **where** \( G(u) \) **is the spectrum of the output signal**, \( F(u) \) **is the spectrum of the input signal**, and \( H(u) \) **is the system transfer function**

- **In many cases, it is easier to analyze an LSI system in the Fourier domain**

- **Forward transform**
  \[ F(u) = \int_{-\infty}^{\infty} f(x)e^{-j2\pi xu} \, dx \]

- **Inverse transform**
  \[ f(x) = \int_{-\infty}^{\infty} F(u)e^{j2\pi xu} \, du \]
Properties for special functions

- $f(x)$ and $F(u)$ are in general, complex functions

- If $f(x)$ real $\rightarrow F(u) = F^*(-u)$
  - **Hermitian**: $\text{Re}[F(u)]$ even, $\text{Im}[F(u)]$ odd

- If $f(x)$ real and even $\rightarrow \text{Im}[F(u)] = 0$ ($F(u)$ is real)

**Forward transform** is the **analysis** of $f(x)$ into its **spectrum** $F(u)$ of sines and cosines at different frequencies, in general, each with a different amplitude and phase.

**Inverse transform** is the **synthesis** of $f(x)$ from $F(u)$. 
### 1-D FOURIER TRANSFORM PAIRS

<table>
<thead>
<tr>
<th>f(x)</th>
<th>F(u)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>δ(u)</td>
</tr>
<tr>
<td>δ(x)</td>
<td>1</td>
</tr>
<tr>
<td>cos(2πu₀x)</td>
<td>$\frac{1}{2</td>
</tr>
<tr>
<td>$\frac{1}{2</td>
<td>x₀</td>
</tr>
<tr>
<td>sin(2πu₀x)</td>
<td>$\frac{j}{2</td>
</tr>
<tr>
<td>rect(x)</td>
<td>sinc(u)</td>
</tr>
<tr>
<td>sinc(x)</td>
<td>rect(u)</td>
</tr>
<tr>
<td>comb(x)</td>
<td>comb(u)</td>
</tr>
<tr>
<td>gaus(x)</td>
<td>gaus(u)</td>
</tr>
<tr>
<td>tri(x)</td>
<td>sinc²(u)</td>
</tr>
</tbody>
</table>
# 1-D Fourier Transform Properties

<table>
<thead>
<tr>
<th>Name</th>
<th>( f(x) )</th>
<th>( F(u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflection</td>
<td>( f(\pm x) )</td>
<td>( F(\pm u) )</td>
</tr>
<tr>
<td>( f(\mp x) )</td>
<td>( F(\mp u) )</td>
<td></td>
</tr>
<tr>
<td>Scaling</td>
<td>( f(x/b) )</td>
<td>(</td>
</tr>
<tr>
<td>Shifting</td>
<td>( f(x \pm x_0) )</td>
<td>( e^{\pm j2\pi x_0 u} F(u) )</td>
</tr>
<tr>
<td></td>
<td>( e^{\pm j2\pi x u_0} f(x) )</td>
<td>( F(u \mp u_0) )</td>
</tr>
<tr>
<td>Derivative</td>
<td>( f^{(k)}(x) )</td>
<td>( (j2\pi u)^k F(u) )</td>
</tr>
<tr>
<td></td>
<td>( (-j2\pi x)^k f(x) )</td>
<td>( F^{(k)}(u) )</td>
</tr>
<tr>
<td>Linearity</td>
<td>( a_1 f_1(x) + a_2 f_2(x) )</td>
<td>( a_1 F_1(u) + a_2 F_2(u) )</td>
</tr>
<tr>
<td>Convolution</td>
<td>( f_1(x) \ast f_2(x) )</td>
<td>( F_1(u)F_2(u) )</td>
</tr>
<tr>
<td></td>
<td>( f_1(x)f_2(x) )</td>
<td>( F_1(u) \ast F_2(u) )</td>
</tr>
</tbody>
</table>
### 1-D FOURIER TRANSFORM PROPERTIES

<table>
<thead>
<tr>
<th>name</th>
<th>f(x)</th>
<th>F(u)</th>
</tr>
</thead>
<tbody>
<tr>
<td>correlation</td>
<td>$f_1(x) \star f_2(x)$</td>
<td>$F_1(u)F_2(-u)$</td>
</tr>
<tr>
<td></td>
<td>$f_1(x)f_2(-x)$</td>
<td>$F_1(u) \star F_2(u)$</td>
</tr>
</tbody>
</table>
LINEAR, SHIFT-INVARIANT (LSI) SYSTEMS (1-D)

Linear: output of a sum of inputs is equal to the sum of the individual outputs

\[ g_1(x) = af_1(x) \ast h(x) \]
\[ g_2(x) = bf_2(x) \ast h(x) \]
\[ g(x) = [af_1(x) + bf_2(x)] \ast h(x) \]
\[ = (af_1(x) \ast h(x) + bf_2(x) \ast h(x)) \]
\[ = g_1(x) + g_2(x) \]

shift-invariant: system response does not change over space

Relation between input \( f(x) \) and output \( g(x) \) of an LSI system

\[ g(x) = f(x) \ast h(x) \]

• where \( h(x) \) is the system impulse response
Fourier transform of LSI system equation

\[ G(u) = F(u)H(u) \]

- where \( H(u) \) is the system transfer function

- \( H(u) \) is a complex filter that modifies the spectrum \( F(u) \) of \( f(x) \)

- In optics, the amplitude of \( H(u) \) is called the Modulation Transfer Function (MTF)

The properties of complex functions give, for an LSI system,

\[ \text{ampl}[G(u)] = \text{ampl}[F(u)] \cdot \text{ampl}[H(u)] \]

\[ \text{phase}[G(u)] = \text{phase}[F(u)] + \text{phase}[H(u)] \]
1-D CASCADED SYSTEMS

N cascaded LSI systems

\[ g(x) = \{ [f(x) \ast h_1(x)] \ast h_2(x) \} \ast \ldots \ast h_N(x) \]
Single system equivalent

\[ h_{net}(x) = h_1(x) \ast h_2(x) \ast \ldots \ast h_N(x) \]

- where \( h_{net} \) is the net system impulse response
FOURIER TRANSFORM EXAMPLES

Ex 1. Find $\text{sinc}(x/2) \ast \text{sinc}(x/3)$

- Convolution in this case is very difficult!

- Take the Fourier transform

$$2\text{rect}(2u) \cdot 3\text{rect}(3u) = 6\text{rect}(3u)$$

- Take the inverse Fourier transform

$$\frac{6}{3}\text{sinc}(x/3) = 2\text{sinc}(x/3)$$
Ex 2. Find spectrum of square wave with a DC bias

- Write square wave as convolution
  \[ f(x) = \frac{1}{5} \text{rect} \left( \frac{x}{2} \right) \ast \text{comb} \left( \frac{x}{5} \right) \]

- Take the Fourier transform
  \[ F(u) = \frac{1}{5} \cdot 2 \cdot 5 \cdot \text{sinc}(2u) \cdot \text{comb}(5u) \]

  \[ = 2\text{sinc}(2u)\text{comb}(5u) \]

  - spectrum is comb function, modulated by sinc function
  - sampled at frequency interval \( \Delta u = 1/5 \), i.e 1/period
  - zeros at \( u = n/2, n = \pm 1, \pm 2, \ldots \)
- if square wave period $P = 2 \times$ pulse width, we have the classic square wave spectrum at
  - $u = 0, \pm 1/P, \pm 3/P, \pm 5/P, \ldots$?

- Let's verify the above statement for an arbitrary period $P$

- $\text{sinc}(2u)$ and $\text{comb}(5u)$

- Graph showing sinc and comb functions.
**sinc(2u) times comb(5u)**

![Graph of sinc(2u) times comb(5u)](image)

Comb, modulated by sinc function
SCALING PROPERTY OF FOURIER TRANSFORMS
Width of $F(u)$ is inversely proportional to width of $f(x)$

- $\text{rect}(x)$
  - Width: $1/2$
  - $F(u)$: $1$

- $\text{rect}(x/2)$
  - Width: $1$
  - $F(u)$: $2\text{sinc}(2u)$

- $\text{rect}(x/3)$
  - Width: $3/2$
  - $F(u)$: $3\text{sinc}(3u)$
\[ F \cos(2\pi x) \]

\[ \delta\delta(u)/2 \]

\[ \cos(4\pi x) \]

\[ \delta\delta(u)/4 \]

\[ \cos(6\pi x) \]

\[ \delta\delta(u)/6 \]
SUPERPOSITION PROPERTY OF FOURIER TRANSFORMS

- Fourier transform of sum of functions equals sum of their individual Fourier transforms

\[ 1 + \cos(2\pi x) \]

\[ \delta(u) + \delta(u/2)/2 \]

\[ \cos(2\pi x) + \cos(4\pi x) \]

\[ \delta(u)/2 + \delta(u/2)/4 \]
SYSTEM ANALYSIS WITH THE FOURIER TRANSFORM

- Variety of applications
- LSI system equation: \( g(x) = f(x) \ast h(x) \)

<table>
<thead>
<tr>
<th>Application</th>
<th>Given</th>
<th>Find</th>
<th>Spatial Domain</th>
<th>Fourier Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>system output</td>
<td>( f(x), h(x) )</td>
<td>( g(x) )</td>
<td>( g(x) = f(x) \ast h(x) )</td>
<td>( G(u) = F(u)H(u) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( g(x) = \mathcal{F}^{-1}[G(u)] )</td>
</tr>
<tr>
<td>system identification</td>
<td>( f(x), g(x) )</td>
<td>( h(x) )</td>
<td>NA</td>
<td>( H(u) = G(u)/F(u) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( h(x) = \mathcal{F}^{-1}[H(u)] )</td>
</tr>
<tr>
<td>inversion</td>
<td>( h(x), g(x) )</td>
<td>( f(x) )</td>
<td>NA</td>
<td>( F(u) = G(u)/H(u) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( f(x) = \mathcal{F}^{-1}[F(u)] )</td>
</tr>
</tbody>
</table>

ill-conditioned
SAMPLING (1-D)

Two components to digitizing an analog signal

- Sample analog signal at discrete values of x
- Quantize the sampled signal amplitude to digital values

Sampling is either ideal (delta function samples) or non-ideal (time-integrated samples)
**IDEAL SAMPLING**

**Sampled function** $f_s(x)$

- **Mathematically**

  $$f_s(x) = \frac{1}{|b|} f(x) \text{comb}\left(\frac{x}{b}\right)$$

- **Sample interval** = $b$, **sample rate** = $1/b$
Fourier domain description

- **Analog signal**  \( f(x) \leftrightarrow F(u) \)

  - **band limit of analog signal** = ± \( u_B \)
  - **(highest frequency component in signal)**
  - **bandwidth of analog signal** = 2\( u_B \)

- **Sampled signal**

  \[
  \frac{1}{|b|} f(x) \text{comb} \left( \frac{x}{b} \right) \leftrightarrow F(u) \ast \text{comb}(bu)
  \]
- **sampling frequency (rate)** = \( u_s = 1/b \)

- **folding frequency** = \( u_f = 1/2b \)
Aliasing occurs where the individual spectra overlap

- **Frequency components above** $u_f$ **appear to be lower frequency components below** $u_f$ **in the sampled signal**

- **Once aliasing occurs, it cannot be removed**
  - **Sometimes, analog signal is pre-filtered (before sampling) by a low-pass filter so that the analog signal has a lower bandwidth and** $u'_B \leq u_f$
Can avoid aliasing if:

- original analog signal is band-limited ($u_B$ is finite) and sample rate satisfies the following condition

$$u_s \geq 2u_B$$

$$\frac{1}{b} \geq 2u_B$$

- or, the sample interval satisfies the condition,

$$b \leq \frac{1}{2u_B}$$